International Journal of Theoretical Physics, Vol. 37, No. 7, 1998

Quantized (1, 0) ⊕ (0, 1) Fields

Valeri V. Dvoeglazov[1]

Received January 7, 1998

We find a mapping between antisymmetric tensor matter fields and the Weinberg $2(2j + 1)$-component "bispinor" fields. Equations which describe the $j = 1$ antisymmetric tensor field coincide with the Hammer–Tucker equations entirely and with the Weinberg ones within a subsidiary condition, the Klein–Gordon equation. A new Lagrangian for the Weinberg theory is proposed which is scalar and Hermitian. It is built on the basis of the concept of 'Weinberg doubles.' The origin of a contradiction between the classical theory, the Weinberg theorem $B - A = \lambda$ for quantum relativistic fields, and the claimed 'longitudity' of the antisymmetric tensor field [transformed on the (1, 0) ⊕ (0, 1) Lorentz group representation] after quantization is clarified. Analogs of the $j = 1/2$ Feynman–Dyson propagator are presented in the framework of the $j = 1$ Weinberg theory. It is then shown that under a definite choice of field functions and initial and boundary conditions the massless $j = 1$ Weinberg–Tucker–Hammer equations contain all the information that the Maxwell equations for the electromagnetic field have. Thus, the former appear to be of use in describing some physical processes.

1. INTRODUCTION

In the 1960s Joos,[1] Weinberg,[2] and Weaver *et al.*[3] proposed a very attractive formalism, called the $2(2j + 1)$ theory, for describing higher-spin particles. For instance, as opposed to the Proca 4-vector potentials which transform according to the (1/2, 1/2) representation of the Lorentz group, in the $j = 1$ case the "bispinor" functions are constructed via the (1, 0) ⊕ (0, 1) representation which is on an equal footing with the description of Dirac $j = 1/2$ particles. The $2(2j + 1)$-component analogs of the Dirac functions in the momentum space were earlier defined as

$$\mathcal{U}_\sigma(\mathbf{p}) = \frac{m}{\sqrt{2}}\begin{pmatrix} D^J(\alpha(\mathbf{p}))\xi_\sigma \\ D^J(\alpha^{-1\dagger}(\mathbf{p}))\xi\sigma \end{pmatrix} \tag{1}$$

[1] Escuela de Física, Universidad Autónoma de Zacatecas, Zacatecas 98068, Zac., Mexico; e-mail: valeri@cantera.reduaz.mx.

0020-7748/98/0700-191 5$15.00/0

for positive-energy states, and

$$\mathcal{V}_\sigma(\mathbf{p}) = \frac{m}{\sqrt{2}} \begin{pmatrix} D^J(\alpha(\mathbf{p})\Theta_{[1/2]})\xi^*_\sigma \\ D^J(\alpha^{-1\dagger}(\mathbf{p})\Theta_{[1/2]})\,(-1)^{2J}\xi^*_\sigma \end{pmatrix} \tag{2}$$

for negative-energy states (e.g., ref. 4, p. 107). The following notations are used:

$$\alpha(\mathbf{p}) = \frac{p_0 + m + (\sigma \cdot \mathbf{p})}{\sqrt{2m(p_0 + m)}}, \qquad \Theta_{[1/2]} = -i\sigma_2 \tag{3}$$

For instance, in the case of spin $j = 1$, one has

$$D^1(\alpha(\mathbf{p})) = 1 + \frac{(\mathbf{J} \cdot \mathbf{p})}{m} + \frac{(\mathbf{J} \cdot \mathbf{p})^2}{m(p_0 + m)} \tag{4a}$$

$$D^1(\alpha^{-1\dagger}(\mathbf{p})) = 1 - \frac{(\mathbf{J} \cdot \mathbf{p})}{m} + \frac{(\mathbf{J} \cdot \mathbf{p})^2}{m(p_0 + m)} \tag{4b}$$

$$D^1(\alpha(\mathbf{p})\Theta_{[1/2]}) = \left[1 + \frac{(\mathbf{J} \cdot \mathbf{p})}{m} + \frac{(\mathbf{J} \cdot \mathbf{p})^2}{m(p_0 + m)}\right]\Theta_{[1]} \tag{4c}$$

$$D^1(\alpha^{-1\dagger}(\mathbf{p})\Theta_{[1/2]}) = \left[1 - \frac{(\mathbf{J} \cdot \mathbf{p})}{m} + \frac{(\mathbf{J} \cdot \mathbf{p})^2}{m(p_0 + m)}\right]\Theta_{[1]} \tag{4d}$$

Here $\Theta_{[1/2]}$, $\Theta_{[1]}$ are the Wigner time-reversal operators for spin 1/2 and 1, respectively. Their forms depend on the choice of the basis for spin matrices. For instance in ref. 14 the spin-1 Wigner operator was chosen in the antidiagonal form. These definitions lead to a formulation in which the physical content given by the positive- and negative-energy "bispinors" is the same (as in the papers of Weinberg and in the further consideration of Tucker and Hammer.[5] In spite of the extensive elaboration of the Weinberg $2(2j + 1)$-component theory since the sixties (*e.g.*, refs. 6–12) such research has not provided new significant insights in particle physics.

However, recently, a physically different construct in the $(1, 0) \oplus (0, 1)$ representation has been proposed.[13] A remarkable feature is that a boson and its antiboson can possess opposite intrinsic parities: "purely by accident, in an attempt to understand an old work of Weinberg[2] and to investigate the possible kinematical origin for the violation of *P*, *CP*, and other discrete symmetries,[14] a Wigner-type quantum field theory[15] was constructed for a spin-one boson."[2] The definition of the negative-energy solutions in this construct is similar to the Dirac construct for the spin-1/2 case:

$$\mathcal{V}_\sigma(\mathbf{p}) = \gamma_5 \mathcal{U}_\sigma(\mathbf{p}) = (-1)^{1-\sigma} S^c_{[1]} \mathcal{U}_{-\sigma}(\mathbf{p}) \tag{5}$$

with $S^c_{[1]}$ being the charge conjugation matrix in the (1, 0) ⊕ (0, 1) representation.[13a,14] They can be built by means of the same procedure used in equations (1) and (2), but taking into account the possibility of an additional phase factor for up (down) components in the bispinorial $j = 1/2$ basis (see, e.g., refs. 14, 17–19).

On the other hand, interest in antisymmetric tensor fields (*e.g.*, refs. 20–27) has long existed and has grown in connection with recent discoveries of tensor couplings in the π^- and K^+ -meson decays. These fields also should transform according to the (1, 0) ⊕ (0, 1) representation.

In the present paper we give a mapping between antisymmetric tensor fields and Weinberg $j = 1$ "bispinors," propose a Lagrangian formalism for a particular model in the (1, 0) ⊕ (0, 1) representation and emphasize consequences relevant to the present situation in fundamental physics. This paper comprises ideas presented in refs. 18 and 28–32.

2. MAPPING BETWEEN ANTISYMMETRIC TENSOR AND WEINBERG FORMULATIONS

Let us begin with the Proca equations for a $j = 1$ massive particle,

$$\partial_\mu F_{\mu\nu} = m^2 A_\nu \tag{6}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{7}$$

in the form given by refs. 16 and 33. The Euclidean metric, $x_\mu = (\mathbf{x}, x_4 = it)$, and the notation $\partial_\mu = (\boldsymbol{\nabla}, -i\partial/\partial t)$, $\partial_\mu^2 = \boldsymbol{\nabla}^2 - \partial_t^2$, are used. By means of the choice of $F_{\mu\nu}$ components as the physical variables one can rewrite the set of equations as

$$m^2 F_{\mu\nu} = \partial_\mu \partial_\alpha F_{\alpha\nu} - \partial_\nu \partial_\alpha F_{\alpha\mu} \tag{8}$$

and

$$\partial_\lambda^2 F_{\mu\nu} = m^2 F_{\mu\nu} \tag{9}$$

[2] Some steps in this direction were made in the sixties,[16] but the authors of those papers did not realize all the possible physical consequences following from their equation.

It is easy to show that they can be represented in the form ($F_{44} = 0$, $F_{4i} = iE_i$, and $F_{jk} = \varepsilon_{jki} B_i$; $p_\alpha = -i\partial_\alpha$):

$$\begin{cases} (m^2 + p_4^2)\, E_i + p_i p_j E_j + i\varepsilon_{ijk} p_4 p_j B_k = 0 \\ (m^2 + \mathbf{p}^2)\, B_i - p_i p_j B_j + i\varepsilon_{ijk} p_4 p_j E_k = 0 \end{cases} \tag{10}$$

or

$$\begin{cases} [m^2 + p_4^2 + \mathbf{p}^2 - (\mathbf{J}\,\mathbf{p})^2]_{ij}\, E_j + p_4\, (\mathbf{J}\,\mathbf{p})_{ij}\, B_j = 0 \\ [m^2 + (\mathbf{J}\,\mathbf{p})^2]_{ij} B_j + p_4\, (\mathbf{J}\,\mathbf{p})_{ij}\, E_j = 0 \end{cases} \tag{11}$$

Adding and subtracting the obtained equations yields

$$\begin{cases} m^2(\mathbf{E} + i\mathbf{B})_i + p_\alpha p_\alpha \mathbf{E}_i - (\mathbf{J}\,\mathbf{p})^2_{ij}\, (\mathbf{E} - i\mathbf{B})_j + p_4\, (\mathbf{J}\mathbf{p})_{ij}\, (\mathbf{B} + i\mathbf{E})_j = 0 \\ m^2(\mathbf{E} - i\mathbf{B})_i + p_\alpha p_\alpha \mathbf{E}_i - (\mathbf{J}\,\mathbf{p})^2_{ij}\, (\mathbf{E} + i\mathbf{B})_j + p_4\, (\mathbf{J}\,\mathbf{p})_{ij}\, (\mathbf{B} - i\mathbf{E})_j = 0 \end{cases} \tag{12}$$

with $(\mathbf{J}_i)_{jk} = -i\varepsilon_{ijk}$ being the $j = 1$ spin matrices. The equations are equivalent (within a constant factor) to the Hammer–Tucker equation[(5)] (see also refs. 11 and 7)

$$(\gamma_{\alpha\beta} p_\alpha p_\beta + p_\alpha p_\alpha + 2m^2)\psi_1 = 0 \tag{13}$$

in the case of the choice $\chi = \mathbf{E} + i\mathbf{B}$ and $\varphi = \mathbf{E} - i\mathbf{B}$, $\psi_1 = \text{column}(\chi, \varphi)$. Matrices $\gamma_{\alpha\beta}$are the covariantly defined matrices of Barut *et al.*[(34)] The equation (13) for massive particles is characterized by positive- and negative-energy solutions with a physical dispersion $E_p = \pm\sqrt{\mathbf{p}^2 + m^2}$; the determinant is equal to

$$\text{Det}\, [\gamma_{\alpha\beta} p_\alpha p_\beta + p_\alpha p_\alpha + 2m^2] = -64 m^6 (p_0^2 - \mathbf{p}^2 - m^2)^3 \tag{14}$$

but some points concerned with a massless limit should be clarified properly.[3]

[3] Questions of the correct relativistic dispersion relations of different $j = 1$ equations (both massive and massless) and of particle interpretations of these solutions were also discussed in ref. 35b. For instance, it was shown that the Maxwell equations possess 'acausal' solution with the energy $E = 0$ and the Weinberg equation, which has common solutions with the Maxwell equations, does *not* reduce entirely to the set of Maxwell equations in the massless limit. Weinberg felt some dissatisfaction when discussing this question [see the first line after (4.21), (4.22) of ref. 2b]; but he failed to indicate in a clear manner that the matrix $(\mathbf{J} \cdot \mathbf{p})$ has no inverse. Several groups have proposed interpretations of the $E = 0$ solution. One can be connected with the 'action-at-a-distance' concept. If one accepts this viewpoint, the electromagnetic field probably has an essentially nonlocal origin and is connected with the structure of space-time itself. But the question of the possibility of experimental observation of such 'action-at-a-distance' is obviously nontrivial, even from a conceptual viewpoint. So this comment is rather speculative.

Following the analysis of ref. 35b, p. 1972,[4] and in accordance with the Dirac technique for obtaining wave equations,[(36)] one can conclude that other equations with a physical dispersion can be obtained from

$$(\gamma_{\alpha\beta} p_\alpha p_\beta + a p_\alpha p_\alpha + b m^2)\psi = 0 \tag{15}$$

with a and b being some numerical constants. As a result of taking into account $E^2 - \mathbf{p}^2 = m^2$, we conclude that an infinite number of equations with the appropriate dispersion exists provided that b and a are connected as follows:

$$\frac{b}{a+1} = 1 \qquad \text{or} \qquad \frac{b}{a-1} = 1$$

However, there are only two equations which do not have 'acausal' solutions. The second one (with $a = -1$ and $b = -2$) is[5]

$$(\gamma_{\alpha\beta} p_\alpha p_\beta - p_\alpha p_\alpha - 2m^2)\psi_2 = 0 \tag{16}$$

Thus, we have found the 'double' of the Hammer–Tucker equation. In tensor form it leads to equations which are dual to (10)

$$\begin{cases} (m^2 + \mathbf{p}^2)\, C_i - p_i p_j C_j - i\varepsilon_{ijk} p_4 p_j D_k = 0 \\ (m^2 + p_4^2)\, D_i + p_i p_j D_j - i\varepsilon_{ijk} p_4 p_j C_k = 0 \end{cases} \tag{17}$$

They can be rewritten in the form [*cf.* equation (8)]

$$m^2 \tilde{F}_{\mu\nu} = \partial_\mu \partial_\alpha \tilde{F}_{\alpha\nu} - \partial_\nu \partial_\alpha \tilde{F}_{\alpha\mu} \tag{18}$$

with $\tilde{F}_{4i} = iD_i$ and $\tilde{F}_{jk} = -\varepsilon_{jki} C_i$. The vector C_i is an analog of E_i, and D_i is an analog of B_i, because in some cases it is convenient to equate $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}$, $\varepsilon_{1234} = -i$. We have used the following properties of the antisymmetric Levi-Civita tensor:

$$\varepsilon_{ijk}\varepsilon_{ijl} = 2\delta_{kl}, \qquad \varepsilon_{ijk}\varepsilon_{ilm} = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})$$

$$\varepsilon_{ijk}\varepsilon_{lmn} = \mathrm{Det}\begin{pmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{pmatrix}$$

[4] I mean that some fraction of the operator $\delta_{\alpha\beta} p_\alpha p_\beta$ acting on physically permittable states can be substituted as $m^2 \leftrightarrow -\delta_{\alpha\beta} p_\alpha p_\beta$. The general equation can also be obtained by setting up the generalized Ryder–Burgard relation.[(13,18,19)]

[5] The determinant of the matrix on the left-hand side of the following equation is also given by formula (14).

Comparing the structure of the Weinberg equation ($a = 0$, $b = 1$) with the Hammer–Tucker 'doubles,' one can convince oneself that the former can be represented in a tensor form

$$m^2 F_{\mu\nu} = \partial_\mu \partial_\alpha F_{\alpha\nu} - \partial_\nu \partial_\alpha F_{\alpha\mu} + \frac{1}{2}(m^2 - \partial_\lambda^2) F_{\mu\nu} \tag{19}$$

that corresponds to equation (21). However, as we learnt, it is possible to build an equation, a 'double,'

$$m^2 \tilde{F}_{\mu\nu} = \partial_\mu \partial_\alpha \tilde{F}_{\alpha\nu} - \partial_\nu \partial_\alpha \tilde{F}_{\alpha\mu} + \frac{1}{2}(m^2 - \partial_\lambda^2) \tilde{F}_{\mu\nu} \tag{20}$$

that corresponds to equation (22). The set of Weinberg equations is written in the form

$$(\gamma_{\alpha\beta} p_\alpha p_\beta + m^2)\psi_1 = 0 \tag{21}$$

$$(\gamma_{\alpha\beta} p_\alpha p_\beta - m^2)\psi_2 = 0 \tag{22}$$

Thanks to the Klein–Gordon equation (9) these equations are equivalent to the Proca tensor equations (and to the Hammer–Tucker ones) in the free case. However, if the interaction is included, one cannot say this. Thus, the general solution describing the $j = 1$ states can be presented as a superposition

$$\Psi^{(1)} = c_1 \psi_1^{(1)} + c_2 \psi_2^{(1)} \tag{23}$$

where the constants c_1 and c_2 are to be defined from the boundary, initial, and normalization conditions. Let me note a surprising fact: while both the massive Proca equations (or the Hammer–Tucker ones) and the Klein–Gordon equation do not possess "nonphysical" solutions, their sum, (19), (20), or the Weinberg equations (21), (22), acquire tachyonic solutions. Next, equations (21) and (22) can recast in another form (index T denotes the transpose matrix):

$$[\gamma_{44} p_4^2 + 2\gamma_{4i}^T p_4 p_i + \gamma_{ij} p_i p_j - m^2]\, \psi_1^{(2)} = 0 \tag{24}$$

$$[\gamma_{44} p_4^2 + 2\gamma_{4i}^T p_4 p_i + \gamma_{ij} p_i p_j + m^2]\, \psi_2^{(2)} = 0 \tag{25}$$

respectively, if we understand $\psi_1^{(2)} \sim$ column $(B_i + iE_i, B_i - iE_i) = i\gamma_5\gamma_{44}\psi_1^{(1)}$ and $\psi_2^{(2)} \sim$column $(D_i + iC_i, D_i - iC_i) = i\gamma_5\gamma_{44}\psi_2^{(1)}$. The general solution is again a linear combination

$$\Psi^{(2)} = c_1 \psi_1^{(2)} + c_2 \psi_2^{(2)} \tag{26}$$

From, *e.g.*, equation (21), dividing $\psi_1^{(1)}$ into longitudinal and transverse parts, one finds the equations

$$[E^2 - \mathbf{p}^2]\,(\mathbf{E} + i\mathbf{B})^{\parallel} - m^2\,(\mathbf{E} - i\mathbf{B})^{\parallel}$$
$$+[E^2 + \mathbf{p}^2 - 2E\,(\mathbf{J}\mathbf{p})]\,(\mathbf{E} + i\mathbf{B})^{\perp} - m^2\,(\mathbf{E} - i\mathbf{B})^{\perp} = 0 \qquad (27)$$

and

$$[E^2 - \mathbf{p}^2]\,(\mathbf{E} - i\mathbf{B})^{\parallel} - m^2\,(\mathbf{E} + i\mathbf{B})^{\parallel}$$
$$+ [E^2 + \mathbf{p}^2 + 2E\,(\mathbf{J}\mathbf{p})]\,(\mathbf{E} - i\mathbf{B})^{\perp} - m^2\,(\mathbf{E} + i\mathbf{B})^{\perp} = 0 \qquad (28)$$

One can see that in the classical field theory antisymmetric tensor matter fields are fields with transverse components in the massless limit. The longitudinal "parts" of the above equations do not contain terms $(\mathbf{J}\cdot\mathbf{p})$ provided that the longitudinal modes are associated with the plane waves, too. This can be easily seen on choosing the spin basis where $(S^i)_{jk} = -i\varepsilon^{ijk}$ and using the definition of the longitudinal modes, $\mathbf{p} \times (\mathbf{E} \pm i\mathbf{B})^{\parallel} \equiv 0$.

In connection with the above discussion, the statements of the "longitudinal nature" of the antisymmetric tensor field after quantization made by several authors[22,23,25,28a] are very surprising. For instance, M. Kalb and P. Ramond claimed explicitly (ref. 23c, p. 2283, third line from below) "thus, the massless $\phi_{\mu\nu}$ has one degree of freedom." If this interpretation is used when the Kalb–Ramond Lagrangian describes the (1, 0) ⊕ (0, 1) field only, these authors contradict the correspondence principle and the principles of the relativistic theory. We discuss this question below and in subsequent publications. At the least they had to explain whether the gauge field associated with the Kalb–Ramond gauge invariance has physical significance or not, and what are the transformation laws for the antisymmetric tensor field of the third rank and for "potentials" $\phi_{\mu\nu}$.

Under the transformations $\psi_1^{(1)} \to \gamma_5\psi_2^{(1)}$ or $\psi_1^{(2)} \to \gamma_5\psi_2^{(2)}$ the set of equations (21) and (22), or (24) and (25), remains invariant. The origin of this fact is the dual invariance of the set of the Proca equations. In matrix form, dual transformations correspond to the chiral transformations (for discussion, see, *e.g.*, ref. 37).

Another equation has been proposed in refs. 16 and 13

$$(\gamma_{\alpha\beta}p_\alpha p_\beta + \wp_{u,v}m^2)\psi = 0 \qquad (29)$$

where $\wp_{u,v} = i(\partial/\partial t)/E$, which distinguishes u- (positive-energy) and v- (negative-energy) solutions. For instance, in ref. 13a, footnote 4, it is claimed that

$$\psi_\sigma^+(x) = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2\omega_p}\, u_\sigma(\mathbf{p})\, e^{ipx} \qquad (30)$$

($\omega_p = \sqrt{m^2 + \mathbf{p}^2}$, $p_\mu x_\mu = \mathbf{px} - Et$) must be described by equation (21), and

$$\psi_\sigma^-(x) = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2\omega_p}\, v_\sigma(\mathbf{p})e^{-ipx} \tag{31}$$

by equation (22). Nevertheless, calculating the determinants (14) of equations (13) and (16), we see that the first one has the *negative-energy* solutions and the second one the *positive-energy* solutions. The same is true for both Weinberg equations; they also have these solutions, and below we will give their explicit forms. The question of the choice of appropriate equations for different physical systems was discussed in refs. 14, 17, and 18. The answer depends on desirable particle properties with respect to discrete symmetries.

Let me consider the question of the 'double' solutions on the basis of spinorial analysis. In ref. 16a, p. 1305 (see also ref. 38, pp. 60–61) relations between the Weinberg $j = 1$ "bispinor" (indeed, bivector) and symmetric spinors of $2j$ rank have been discussed. It was noted that "The wave function may be written in terms of two three-component functions ψ = column ($\chi \quad \varphi$), that, for the continuous group, transform independently each of other and that are related to two symmetric spinors:

$$\chi_1 = \chi_{11}, \qquad \chi_2 = \sqrt{2}\chi_{12}, \qquad \chi_3 = \chi_{22} \tag{32}$$

$$\varphi_1 = \varphi^{11}, \qquad \varphi_2 = \sqrt{2}\varphi^{12}, \qquad \varphi_3 = \varphi^{22} \tag{33}$$

when the standard representation for the spin-one matrices, with S_3 diagonal is used." Under the inversion operation we have the following rules (ref. 38, p. 59): $\varphi^\alpha \to \chi_{\dot\alpha}$, $\chi_{\dot\alpha} \to \varphi^\alpha$, $\varphi_\alpha \to -\chi^{\dot\alpha}$, and $\chi^{\dot\alpha} \to -\varphi_\alpha$. Hence, one can deduce (if one takes $\chi_{\dot\alpha\dot\beta} = \chi_{\{\dot\alpha}\chi_{\dot\beta\}}$, $\varphi^{\alpha\beta} = \varphi^{\{\alpha}\varphi^{\beta\}}$)

$$\chi_{11} \to \varphi^{11}, \qquad \chi_{22} \to \varphi^{22}, \qquad \chi_{\{12\}} \to \varphi^{\{12\}} \tag{34}$$

$$\varphi^{11} \to \chi_{11}, \qquad \varphi^{22} \to \chi_{22}, \qquad \varphi^{\{12\}} \to \chi_{\{12\}} \tag{35}$$

However, this definition of symmetric spinors of the second rank χ and φ is ambiguous. We are also able to define, *e.g.*, $\tilde{\chi}_{\dot\alpha\dot\beta} = \chi_{\{\dot\alpha}H_{\dot\beta\}}$ and $\tilde{\varphi}^{\alpha\beta} = \varphi^{\{\alpha}\Phi^{\beta\}}$, where $H_{\dot\beta} = \varphi^*_\beta$, $\Phi^\beta = (\chi^{\dot\beta})^*$. It is straightforward to show that in the framework of the second definition we have under the space-inversion operation

$$\tilde{\chi}_{11} \to -\tilde{\varphi}^{11}, \qquad \tilde{\chi}_{22} \to -\tilde{\varphi}^{22}, \qquad \tilde{\chi}_{\{12\}} \to -\tilde{\varphi}^{\{12\}} \tag{36}$$

$$\tilde{\varphi}^{11} \to -\tilde{\chi}_{11}, \qquad \tilde{\varphi}^{22} \to -\tilde{\chi}_{22}, \qquad \tilde{\varphi}^{\{12\}} \to -\tilde{\chi}_{\{12\}} \tag{37}$$

The Weinberg "bispinor" ($\chi_{\dot\alpha\dot\beta} \; \varphi^{\alpha\beta}$) corresponds to equations (24) and (25), and ($\tilde{\chi}_{\dot\alpha\dot\beta} \; \tilde{\varphi}^{\alpha\beta}$), to equations (21) and (22). Similar conclusions can be achieved in the case of the parity definition as $P^2 = -1$. Transformation rules are then $\varphi^{\alpha} \to i\chi_{\dot{\alpha}}$, $\chi_{\dot{\alpha}} \to i\varphi^{\alpha}$, $\varphi_{\alpha} \to -i\chi^{\dot{\alpha}}$, and $\chi^{\dot{\alpha}} \to -i\varphi_{\alpha}$ (ref. 38, p. 59). Hence, $\chi_{\dot{\alpha}\dot{\beta}} \leftrightarrow -\varphi^{\alpha\beta}$ and $\tilde{\chi}_{\dot{\alpha}\dot{\beta}} \leftrightarrow -\tilde{\varphi}^{\alpha\beta}$, but $\varphi_{\alpha}^{\ \beta} \leftrightarrow \chi_{\beta}^{\ \alpha}$ and $\tilde{\varphi}_{\alpha}^{\ \beta} \leftrightarrow \tilde{\chi}_{\beta}^{\ \alpha}$.

In previous formulations of the Weinberg theory the following Lagrangian was proposed[(8,9,11b,28a,b)]:

$$\mathcal{L}^{W} = -\partial_{\mu}\overline{\psi}\gamma_{\mu\nu}\partial_{\nu}\psi - m^{2}\overline{\psi}\psi \tag{38}$$

$\gamma_{\mu\nu}$ are the Barut–Muzinich–Williams matrices, which are chosen to be Hermitian. It is scalar[(28a)] and Hermitian *cf.* ref. 8[6] and it contains only first-order time derivatives. Again taking the interpretation of the "6-spinor" as[7]

$$\begin{cases} \chi = \mathbf{E} + i\mathbf{B} \\ \phi = \mathbf{E} - i\mathbf{B} \end{cases} \tag{39}$$

where ψ = column $(\chi\ \phi)$ and $\mathbf{E}$ and $\mathbf{B}$ are real 3-vectors, we can rewrite the Lagrangian (38) in the following way:

$$\begin{aligned} \mathcal{L}^{AT} = &-(\partial_{\mu}F_{\nu\alpha})(\partial_{\mu}F_{\nu\alpha}) + 2(\partial_{\mu}F_{\mu\alpha})(\partial_{\nu}F_{\nu\alpha}) \\ &+ 2(\partial_{\mu}F_{\nu\alpha})(\partial_{\nu}F_{\mu\alpha}) + m^{2}F_{\mu\nu}F_{\mu\nu} \end{aligned} \tag{40}$$

In the massless limit this form of the Lagrangian leads to the Euler–Lagrange equation

$$(\square - m^{2})F_{\alpha\beta} - 2(\partial_{\beta}F_{\alpha\mu,\mu} - \partial_{\alpha}F_{\beta\mu,\mu}) = 0 \tag{41}$$

where $\square = \partial_{\nu}\partial_{\nu}$. After the application of the generalized Lorentz condition[(22)] the massless Lagrangian (40) becomes equivalent to the Lagrangian of a free massless skew-symmetric field given in ref. 22:

$$\mathcal{L}^{H} = \frac{1}{8} F_{k}F_{k} \tag{42}$$

with $F_{k} = i\epsilon_{kjmn}F_{jm,n}$. It is rewritten as $(m = 0)$

$$\begin{aligned} \mathcal{L}^{H} &= -\frac{1}{4}(\partial_{\mu}F_{\nu\alpha})(\partial_{\mu}F_{\nu\alpha}) + \frac{1}{2}(\partial_{\mu}F_{\nu\alpha})(\partial_{\nu}F_{\mu\alpha}) \\ &= \frac{1}{4}\mathcal{L}^{AT} - \frac{1}{2}(\partial_{\mu}F_{\mu\alpha})(\partial_{\nu}F_{\nu\alpha}) \end{aligned} \tag{43}$$

[6] When the Euclidean metric is used we take $\partial_{\mu}^{\dagger} = (\nabla, -\partial/\partial x_{4})$, provided that $\partial_{\mu} = (\nabla, \partial/\partial x_{4})$.[(33)]

[7] One can also choose

$$\psi^{(2)} = \begin{pmatrix} \mathbf{E} + i\mathbf{B} \\ -\mathbf{E} + i\mathbf{B} \end{pmatrix} = \gamma_{5}\psi$$

Since $\overline{\psi}^{(2)} = -\overline{\psi}\gamma_{5}$ the dynamical term (38) is not changed. But the sign in the mass term is inverse.

which proves the statement made above. After the application of the Fermi method *mutatis mutandis* as in ref. 22 (*cf.* with the quantization procedure for a 4-vector potential field) one finds that the Lagrangians (38) and (42) describe massless particles possessing longitudinal physical components only. Transverse components are removed by means of the "gauge" transformation

$$F_{\mu\nu} \rightarrow F_{\mu\nu} + A_{[\mu\nu]} = F_{\mu\nu} + \partial_\nu \Lambda_\mu - \partial_\mu \Lambda_\nu \tag{44}$$

(or by a transformation similar to the above, but applied to the Weinberg bivector). This is a contradiction noted in refs. 28a and 28b: the $j = 1$ antisymmetric tensor field was believed to possess the longitudinal component only, and the helicity is therefore equal to $\lambda = 0$. Yet they transform according to the $(1, 0) + (0, 1)$ representation of the Lorentz group (like a Helmholtz–Weinberg bivector). How is the Weinberg theorem [(2)] for the (A, B) representation to be treated in this case?[8] If we want to have well-defined creation and annihilation operators the antisymmetric tensor field should have helicities $\lambda = \pm 1$.[9] Moreover, do the claims of the "longitudinal nature" of the antisymmetric tensor field and hence the Weinberg $j = 1$ field signify that we must abandon the correspondence principle? In classical physics we know that an antisymmetric tensor field has transverse components; see also (27) and (28).[10]

This contradiction has been analyzed in refs. 29–32, 39, 40 in detail. The result is that transverse components are always linked with longitudinal spin components and can be decoupled only in particular cases. Using the Weinberg formalism, we provide additional support to this conclusion in the following section.

We conclude this section: both the theory of Ahluwalia *et al.*[(13,14)] and the model based on the use of ψ_1 and ψ_2 are connected with the antisymmetric tensor matter field description. They have to be quantized consistently. Special attention should be paid to the translational and rotational invariance (in fact the conservation of energy–momentum and angular momentum), the interaction representation, causality, locality, and covariance of the theory, *i.e.*, to all topics which are axioms of the modern quantum field theory.[(41,42)] A consistent theory also has to take into account the degeneracy of states: two dual functions ψ_1 and ψ_2 (or $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$, the 'doubles') are considered to yield the same spectrum.

[8] *Weinberg theorem*: The fields constructed from the massless particle operator $a(\mathbf{p}, \lambda)$ of definite helicity transform according to the representation (A, B) such that $B - A = \lambda$.

[9] Several authors have indicated this from different viewpoints.[(20,21,24,13)]

[10] Concerning this contradiction, one of the referees commented, "the contradictions following equation (44) arise from a confusion regarding the 'transverse' nature of the **E** and **B** fields with respect to the **p** and at the same time 'longitudinal' nature of helicity for the SAME **E** and **B** configuration." I agree.

3. WHAT PARTICLES ARE DESCRIBED BY THE WEINBERG THEORY?

In the previous section the concept of the Weinberg $j = 1$ field as a system of degenerate states has been proposed. As a matter of fact, a model with the Weinberg 'doubles' is equivalent to dual electrodynamics with the antisymmetric tensor field $F_{\mu\nu}$ and its dual $\tilde{F}_{\mu\nu}$. Unfortunately, many works concerned with dual theories(24,37,43,44) did not examine quantization issues in detail and many specific features have not been taken into account.[11]

We begin with a Lagrangian which is similar to (38), but includes additional terms which respond to the Weinberg 'double'[12]:

$$\mathcal{L}^{(1)} = -\partial_\mu\overline{\psi}_1\gamma_{\mu\nu}\partial_\nu\psi_1 - \partial_\mu\overline{\psi}_2\gamma_{\mu\nu}\partial_\nu\psi_2 - m^2\overline{\psi}_1\psi_1 + m^2\overline{\psi}_2\psi_2 \tag{45}$$

The Lagrangian (45) leads to equations (21), (22), which possess solutions with a "correct" bradyon physical dispersion and tachyonic solutions as well. The second equation coincides with the Ahluwalia *et al.* equation for v spinors [equation (13), ref. 13a] or with (12) of ref. 16c. If one accepts the concept of the Weinberg field as a set of degenerate states, one has to allow for possible transitions $\psi_1 \leftrightarrow \psi_2$ (or $F_{\mu\nu} \leftrightarrow \tilde{F}_{\mu\nu}$). At first sight, one can propose the Lagrangian with the following dynamical part:

$$\mathcal{L}^{(2')} = -\partial_\mu\overline{\psi}_1\gamma_{\mu\nu}\partial_\nu\psi_2 - \partial_\mu\overline{\psi}_2\gamma_{\mu\nu}\partial_\nu\psi_1 \tag{46}$$

where ψ_1 and ψ_2 are defined by equations (21), (22). But this form appears not to admit a mass term in the usual manner. From a mathematical viewpoint one can find a solution: set m^2 to be a pure imaginary quantity (or in the operator formulation, the anti-Hermitian operator). We touched upon this case earlier.(30) A more logical approach seems to be to regard all four states described by (21), (22), (24), (25). The following Lagrangian can be proposed in this case:

$$\begin{aligned}\mathcal{L}^{(2)} = &-\partial_\mu\psi_1^{(1)\dagger}\tilde{\gamma}_{\mu\nu}\partial_\nu\psi_2^{(2)} - \partial_\mu\psi_2^{(2)\dagger}\gamma_{\mu\nu}\partial_\nu\psi_1^{(1)}\\ &- \partial_\mu\psi_2^{(1)\dagger}\,\tilde{\gamma}_{\mu\nu}\partial_\nu\psi_1^{(2)} - \partial_\mu\psi_1^{(2)\dagger}\,\gamma_{\mu\nu}\partial_\nu\psi_2^{(1)}\\ &- m^2\psi_2^{(2)\dagger}\,\psi_1^{(1)} - m^2\psi_1^{(1)\dagger}\,\psi_2^{(2)} + m^2\psi_2^{(1)\dagger}\,\psi_1^{(2)} + m^2\,\psi_1^{(2)\dagger}\,\psi_2^{(1)}\end{aligned} \tag{47}$$

Both the Lagrangians (45) and (47) are scalars[13] and Hermitian and they

[11] Dual formulations of the Dirac field, the (1/2, 0) ⊕ (0, 1/2) representation, have also been considered, *e.g.*, refs. 45–47, 17, 18. The interaction of the Dirac field with the dual fields $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ has been considered in ref. 48 (this implies the existence of the anomalous electric dipole moment of a fermion).

[12] Of course, one can use another form with the substitutions $\psi_{1,2}^{(1)} \to \psi_{2,1}^{(2)}$ and $\gamma_{\mu\nu} \to \tilde{\gamma}_{\mu\nu}$, where $\gamma_{\mu\nu} \equiv \gamma_{\mu\nu}^T \equiv \gamma_{44}\gamma_{\mu\nu}\gamma_{44}$.

[13] It is easy to verify this by taking into account the proposed interpretations of $\psi_i^{(k)}(x)$ which are connected with the tensor $F^{\mu\nu}$ and its dual. There is also another way, γ, the use of explicit forms of momentum-space "6-spinors"; see below.

contain only first-order time derivatives. They both lead to similar equations for $\psi_1^{(1,2)}(x)$ and $\psi_2^{(1,2)}(x)$, but one should not forget the difference in signs in mass terms when considering the equations for $\psi_i^{(k)}(x)$.

At this point I would like to regard the question of solutions in momentum space. Using the plane-wave expansion[14]

$$\psi_1^{(k)}(x) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3} \frac{1}{m\sqrt{2E_p}} \times \left[\mathcal{U}_1^{(k)\sigma}(\mathbf{p}) a_\sigma^{(k)}(\mathbf{p}) e^{ipx} + \mathcal{V}_1^{(k)\sigma}(\mathbf{p}) b_\sigma^{(k)\dagger}(\mathbf{p}) e^{-ipx} \right] \quad (48)$$

$$\psi_2^{(k)}(x) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3} \frac{1}{m\sqrt{2E_p}} \times \left[\mathcal{U}_2^{(k)\sigma}(\mathbf{p}) c_\sigma^{(k)}(\mathbf{p}) e^{ipx} + \mathcal{V}_2^{(k)\sigma}(\mathbf{p}) d_\sigma^{(k)\dagger}(\mathbf{p}) e^{-ipx} \right] \quad (49)$$

($E_p = \sqrt{\mathbf{p}^2 + m^2}$), one can see that the momentum-space 'double' equations

$$\left[-\gamma_{44}E^2 + 2iE\gamma_{4i}\mathbf{p}_i + \gamma_{ij}\mathbf{p}_i\mathbf{p}_j + m^2 \right] \mathcal{U}_1^\sigma(\mathbf{p}) = 0 \quad (\text{or } \mathcal{V}_1^\sigma(\mathbf{p})) \quad (50)$$

$$\left[-\gamma_{44}E^2 + 2iE\gamma_{4i}\mathbf{p}_i + \gamma_{ij}\mathbf{p}_i\mathbf{p}_j - m^2 \right] \mathcal{U}_2^\sigma(\mathbf{p}) = 0 \quad (\text{or } \mathcal{V}_2^\sigma(\mathbf{p})) \quad (51)$$

are satisfied by "bispinors"

$$\mathcal{U}_1^{(1)\sigma}(\mathbf{p}) = \frac{m}{\sqrt{2}} \begin{pmatrix} \left[1 + \frac{(\mathbf{J}\mathbf{p})}{m} + \frac{(\mathbf{J}\mathbf{p})^2}{m(E+m)} \right] \xi_\sigma \\ \left[1 - \frac{(\mathbf{J}\mathbf{p})}{m} + \frac{(\mathbf{J}\mathbf{p})^2}{m(E+m)} \right] \xi_\sigma \end{pmatrix} \quad (52)$$

[14] I stress that my present aim is to keep the mathematical approach as general as possible. The relevance of different photon spin states to different forms of field operators will be studied in more detail in forthcoming publications.

and

$$\mathcal{U}_2^{(1)\sigma}(\mathbf{p}) = \frac{m}{\sqrt{2}} \begin{pmatrix} \left[1 + \frac{(\mathbf{Jp})}{m} + \frac{(\mathbf{Jp})^2}{m(E+m)}\right]\xi_\sigma \\ \left[-1 + \frac{(\mathbf{Jp})}{m} - \frac{(\mathbf{Jp})^2}{m(E+m)}\right]\xi_\sigma \end{pmatrix} \tag{53}$$

respectively. The form (52) has been presented by Hammer and Tucker[5] and Novozhilov[4] (see also ref. 11). The bispinor normalization in the cited papers is chosen to be unity. However, as mentioned in ref. 13 it is more convenient to work with bispinors normalized to the mass, *e.g.*, $\pm m^{2j}$, in order to make zero-momentum spinors vanish in the massless limit. Here and below I keep the normalization of bispinors as in ref. 13. Bispinors of Ahluwalia *et al.*[13] can be written in the more compact form

$$u^\sigma_{\mathrm{AJG}}(\mathbf{p}) = \begin{pmatrix} \left[m + \frac{(\mathbf{Jp})^2}{E+m}\right]\xi_\sigma \\ (\mathbf{Jp})\xi_\sigma \end{pmatrix}, \qquad v^\sigma_{\mathrm{AJG}}(\mathbf{p}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u^\sigma_{\mathrm{AJG}}(\mathbf{p}) \tag{54}$$

They coincide with the Hammer–Tucker–Novozhilov bispinors within a normalization and a unitary transformation by the $\mathcal{U}$ matrix:

$$u^\sigma_{[13]}(\mathbf{p}) = m \cdot \mathbf{U}\mathcal{U}^\sigma_{[5,4]}(\mathbf{p}) = \frac{m}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathcal{U}^\sigma_{[5,4]}(\mathbf{p}) \tag{55}$$

$$v^\sigma_{[13]}(\mathbf{p}) = m \cdot \mathbf{U}\gamma_5\, \mathcal{U}^\sigma_{[5,4]}(\mathbf{p}) = \frac{m}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \gamma_5\, \mathcal{U}^\sigma_{[5,4]}(\mathbf{p}) \tag{56}$$

But, as we have found, the Weinberg equations (with $+m^2$ and with $-m^2$) have solutions with both positive and negative energies. In the framework of this paper one can consider that $\mathcal{V}^{(1,2)}_\sigma(\mathbf{p}) = (-1)^{1-\sigma}\gamma_5 S^c_{[1]}\, \mathcal{U}^{(1,2)}_{-\sigma}(\mathbf{p})$ and, thus, the explicit form of the negative-energy solutions would be the same as of the positive-energy solutions in accordance with definitions (1), (2); see the discussion in Section 1. Thus, in the case of the choice of $\mathcal{U}_1^{(1)\sigma}(\mathbf{p})$ and $\mathcal{V}_2^{(1)\sigma} \sim \gamma_5 \mathcal{U}_1^{(1)\sigma}(\mathbf{p})$ as physical bispinors we come to the Bargmann–Wightman–Wigner-type (BWW) quantum field model proposed by Ahluwalia *et al.* Of course, following the same logic, one can choose

$\mathcal{U}_2^{(1)\sigma}$ and $\mathcal{V}_1^{(1)\sigma}$ bispinors and come to yet another version of the BWW theory. While in this case the parities of a boson and its antiboson are opposite, we have -1 for $\mathcal{U}$-bispinor and $+1$ for $\mathcal{V}$-bispinor, *i.e.*, different in sign from the model of Ahluwalia *et al.*[15] The construct proposed by Weinberg[(2)] and developed in this paper is also possible. I do not agree with the claim of the authors of ref. 13a (footnote 4) that $\mathcal{V}_1^{(1)\sigma}(\mathbf{p})$ are not solutions of equation (21). The origin of the possibility that the $\mathcal{U}_i$- and $\mathcal{V}_i$-bispinors in (50), (51) can coincide is that the Weinberg equations are of second order in time derivatives. The Bargmann–Wightman–Wigner construct presented by Ahluwalia *et al.*[(13)] is not the only construct in the $(1, 0) \oplus (0, 1)$ representation and one can start with the earlier definitions of the $2(2j + 1)$ bispinors.

Next, in Section 2 we gave two additional equations (24), (25). Their solutions can also be useful because of the possibility of the use of the Lagrangian form (47). The solutions in the momentum representation are written as follows:

$$\mathcal{U}_1^{(2)\sigma}(\mathbf{p}) = \frac{m}{\sqrt{2}} \begin{pmatrix} \left[1 - \frac{(\mathbf{J}\mathbf{p})}{m} + \frac{(\mathbf{J}\mathbf{p})^2}{m(E+m)}\right]\xi_\sigma \\ \left[-1 - \frac{(\mathbf{J}\mathbf{p})}{m} - \frac{(\mathbf{J}\mathbf{p})^2}{m(E+m)}\right]\xi_\sigma \end{pmatrix} \tag{57}$$

$$\mathcal{U}_2^{(2)\sigma}(\mathbf{p}) = \frac{m}{\sqrt{2}} \begin{pmatrix} \left[1 - \frac{(\mathbf{J}\mathbf{p})}{m} + \frac{(\mathbf{J}\mathbf{p})^2}{m(E+m)}\right]\xi_\sigma \\ \left[1 + \frac{(\mathbf{J}\mathbf{p})}{m} + \frac{(\mathbf{J}\mathbf{p})^2}{m(E+m)}\right]\xi_\sigma \end{pmatrix} \tag{58}$$

Therefore, one has $\mathcal{U}_2^{(1)}(\mathbf{p}) = \gamma_5\mathcal{U}_1^{(1)}(\mathbf{p})$ and $\overline{\mathcal{U}}_2^{(1)}(\mathbf{p}) = -\overline{\mathcal{U}}_1^{(1)}(\mathbf{p})\gamma_5$; $\mathcal{U}_1^{(2)}(\mathbf{p}) = \gamma_5\gamma_{44}\mathcal{U}_1^{(1)}(\mathbf{p})$ and $\overline{\mathcal{U}}_1^{(2)}(\mathbf{p}) = \overline{\mathcal{U}}_1^{(1)}(\mathbf{p})\,\gamma_5\gamma_{44}$; $\mathcal{U}_2^{(2)}(\mathbf{p}) = \gamma_{44}\mathcal{U}_1^{(1)}(\mathbf{p})$ and $\overline{\mathcal{U}}_2^{(2)}(\mathbf{p}) = \overline{\mathcal{U}}_1^{(1)}(\mathbf{p})\gamma_{44}$. In fact, they are connected by transformations of the inversion group.

Let me now repeat the quantization procedure for antisymmetric tensor field presented, *e.g.*, in ref. 22; however, it will be applied to the

[15] At our present level of knowledge this mathematical difference has no physical significance, because it is believed that "ONLY relative intrinsic parities of the particles are physically observable." But we want to stay in the most general framework and perhaps some forms of interactions can lead to observed physical differences between these models.

Weinberg field. Let me trace contributions of $\mathcal{L}^{(1)}$ to dynamical invariants. From the definitions[33]

$$\mathcal{T}_{\mu\nu} = -\sum_i \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\nu \phi_i + \partial_\nu \overline{\phi}_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\phi}_i)} \right\} + \mathcal{L}\delta_{\mu\nu} \tag{59}$$

$$P_\mu = \int \mathcal{P}_\mu(x)\, d^3x = -i \int \mathcal{T}_{4\mu}\, d^3x \tag{60}$$

one can find the energy-momentum tensor[16]

$$\mathcal{T}^{(1)}_{\mu\nu} = \partial_\alpha \overline{\psi}_1 \gamma_{\alpha\mu} \partial_\nu \psi_1 + \partial_\nu \overline{\psi}_1 \gamma_{\mu\alpha} \partial_\alpha \psi_1 + \partial_\alpha \overline{\psi}_2 \gamma_{\alpha\mu} \partial_\nu \psi_2 + \partial_\nu \overline{\psi}_2 \gamma_{\mu\alpha} \partial_\alpha \psi_2 + \mathcal{L}^{(1)} \delta_{\mu\nu} \tag{65}$$

[16] Finding the classical dynamical invariants from the Lagrangian $\mathcal{L}^{(2)}$ does not present any difficulty. They are

$$\begin{aligned}\mathcal{T}^{(2)}_{\mu\nu} = {} & \partial_\alpha \psi_1^{(1)\dagger} \tilde{\gamma}_{\alpha\mu} \partial_\nu \psi_2^{(2)} + \partial_\alpha \psi_1^{(2)\dagger} \gamma_{\alpha\mu} \partial_\nu \psi_2^{(1)} + \partial_\alpha \psi_2^{(1)\dagger} \tilde{\gamma}_{\alpha\mu} \partial_\nu \psi_1^{(2)} + \partial_\alpha \psi_2^{(2)\dagger} \gamma_{\alpha\mu} \partial_\nu \psi_1^{(1)} \\ & + \partial_\nu \psi_1^{(1)\dagger} \tilde{\gamma}_{\mu\alpha} \partial_\alpha \psi_2^{(2)} + \partial_\nu \psi_1^{(2)\dagger} \gamma_{\mu\alpha} \partial_\alpha \psi_2^{(1)} + \partial_\nu \psi_2^{(1)\dagger} \tilde{\gamma}_{\mu\alpha} \partial_\alpha \psi_1^{(2)} + \partial_\nu \psi_2^{(2)\dagger} \gamma_{\mu\alpha} \partial_\alpha \psi_1^{(1)} \\ & + \mathcal{L}^{(2)} \delta_{\mu\nu}\end{aligned} \tag{61}$$

$$\begin{aligned}\mathcal{H}^{(2)} = \int & [-\partial_4 \psi_1^{(1)\dagger} \gamma_{44} \partial_4 \psi_2^{(2)} + \partial_i \psi_1^{(1)\dagger} \gamma_{ij} \partial_j \psi_2^{(2)} - \partial_4 \psi_1^{(2)\dagger} \gamma_{44} \partial_4 \psi_2^{(1)} + \partial_i \psi_1^{(2)\dagger} \gamma_{ij} \partial_j \psi_2^{(1)} \\ & - \partial_4 \psi_2^{(1)\dagger} \gamma_{44} \partial_4 \psi_1^{(2)} + \partial_i \psi_2^{(1)\dagger} \gamma_{ij} \partial_j \psi_1^{(2)} - \partial_4 \psi_2^{(2)\dagger} \gamma_{44} \partial_4 \psi_1^{(1)} + \partial_i \psi_2^{(2)\dagger} \gamma_{ij} \partial_j \psi_1^{(1)} \\ & + m^2 \psi_1^{(1)\dagger} \psi_2^{(2)} - m^2 \psi_1^{(2)\dagger}\, \psi_2^{(1)} - m^2 \psi_2^{(1)\dagger} \psi_1^{(2)\dagger} + m^2 \psi_2^{(2)\dagger} \psi_1^{(1)}]\, d^3x\end{aligned} \tag{62}$$

The charge operator and the spin tensor are

$$\begin{aligned}\mathcal{T}^{(2)}_{\mu} = i[& \partial_\alpha \psi_1^{(1)\dagger} \tilde{\gamma}_{\alpha\mu} \psi_2^{(2)} + \partial_\alpha \psi_1^{(2)\dagger} \gamma_{\alpha\mu}\, \psi_2^{(1)} + \partial_\alpha \psi_2^{(1)\dagger}\, \tilde{\gamma}_{\alpha\mu} \psi_1^{(2)} + \partial_\alpha \psi_2^{(2)\dagger} \gamma_{\alpha\mu} \psi_1^{(1)} \\ & - \psi_1^{(1)\dagger} \tilde{\gamma}_{\mu\alpha} \partial_\alpha\, \psi_2^{(2)} - \psi_1^{(2)\dagger} \gamma_{\mu\alpha} \partial_\alpha \psi_2^{(1)} - \psi_2^{(1)\dagger}\, \tilde{\gamma}_{\mu\alpha} \partial_\alpha \psi_1^{(2)} - \psi_2^{(2)\dagger} \gamma_{\mu\alpha} \partial_\alpha \psi_1^{(1)}]\end{aligned} \tag{63}$$

$$\begin{aligned}S^{(2)}_{\mu\nu,\lambda} = {} & i[\partial_\alpha \psi_1^{(1)\dagger} \tilde{\gamma}_{\alpha\lambda} N_{\mu\nu}^{\psi_2^{(2)}} \psi_2^{(2)} + \partial_\alpha \psi_1^{(2)\dagger} \gamma_{\alpha\lambda} N_{\mu\nu}^{\psi_2^{(1)}} \psi_2^{(1)} + \partial_\alpha \psi_2^{(1)\dagger} \tilde{\gamma}_{\alpha\lambda}\, N_{\mu\nu}^{\psi_1^{(2)}}\, \psi_1^{(2)} \\ & + \partial_\alpha \psi_2^{(2)\dagger}\, \gamma_{\alpha\lambda} N_{\mu\nu}^{\psi_1^{(1)}}\, \psi_1^{(1)} + \psi_1^{(1)\dagger}\, N_{\mu\nu}^{\psi_1^{(1)\dagger}}\, \tilde{\gamma}_{\lambda\alpha} \partial_\alpha \psi_2^{(2)} + \psi_1^{(2)\dagger} N_{\mu\nu}^{\psi_1^{(2)\dagger}} \gamma_{\lambda\alpha} \partial_\alpha \psi_2^{(1)} \\ & + \psi_2^{(1)\dagger}\, N_{\mu\nu}^{\psi_2^{(1)\dagger}} \tilde{\gamma}_{\lambda\alpha} \partial_\alpha\, \psi_{(1)}^{(2)} + \psi_2^{(2)\dagger} N_{\mu\nu}^{\psi_2^{(1)\dagger}} \gamma_{\lambda\alpha} \partial_\alpha \psi_1^{(1)}]\end{aligned} \tag{64}$$

Questions of translational invariance, the choice of bispinors corresponding to the physical states, renormalizability of the theory based on $\mathcal{L}^{(2)}$, and the possibility of the existence of chiral charge for this system [as for the Majorana states in the (1/2, 0) ⊕ (0, 1/2) representation, as shown in previous papers of the author] will be given detailed elaboration in a separate paper.

As a result the Hamiltonian is[17]

$$\mathcal{H}^{(1)} = \int [-\partial_4\overline{\psi}_2\gamma_{44}\partial_4\psi_2 + \partial_i\overline{\psi}_2\gamma_{ij}\partial_j\psi_2 - \partial_4\overline{\psi}_1\gamma_{44}\partial_4\psi_1 + \partial_i\overline{\psi}_1\gamma_{ij}\partial_j\psi_1 + m^2\overline{\psi}_1\psi_1 - m^2\overline{\psi}_2\psi_2]\, d^3x \quad (66)$$

The quantized Hamiltonian

$$\mathcal{H}^{(1)} = \sum_\sigma \int \frac{d^3p}{(2\pi)^3} E_p\, [a_\sigma^\dagger(\mathbf{p})\, a_\sigma(\mathbf{p}) + b_\sigma(\mathbf{p})\, b_\sigma^\dagger(\mathbf{p}) + c_\sigma^\dagger(\mathbf{p})\, c_\sigma(\mathbf{p}) + d_\sigma(\mathbf{p})\, d_\sigma^\dagger(\mathbf{p})] \quad (67)$$

is obtained after using the plane-wave expansion following the procedure of, *e.g.*, refs. 41 and 42. Acknowledging the suggestion of a colleague, I regard the matters of translational invariance and positive-definiteness of the energy in the theory based on the $\mathcal{L}^{(1)}$ in more detail. I proceed step by step to the fermionic consideration of ref. 41, p. 145.[18] The condition of translational invariance imposes the constraints

$$\psi_1(x+a) = e^{-iP_\mu a_\mu}\,\psi_1(x)e^{iP_\mu a_\mu}, \qquad \psi_2(x+a) = e^{-iP_\mu a_\mu}\,\psi_2(x)e^{iP_\mu a_\mu} \quad (68)$$

or, in differential form,

$$\partial_\mu\psi_1(x) = -i[P_\mu, \psi_1(x)]_-, \qquad \partial_\mu\overline{\psi}_1(x) = -i\,[P_\mu, \overline{\psi}_1(x)]_- \quad (69)$$

$$\partial_\mu\psi_2(x) = -i[P_\mu, \psi_2(x)]_-, \qquad \partial_\mu\overline{\psi}_2(x) = -i[P_\mu, \overline{\psi}_2(x)]_- \quad (70)$$

These constraints are satisfied provided that

$$[P_\mu, a_\sigma(\mathbf{p})]_- = -p_\mu a_\sigma(\mathbf{p}), \qquad [P_\mu, b_\sigma(\mathbf{p})]_- = -p_\mu b_\sigma(\mathbf{p}) \quad (71)$$

$$[P_\mu, a_\sigma^\dagger(\mathbf{p})]_- = +p_\mu a_\sigma^\dagger(\mathbf{p}), \qquad [P_\mu, b_\sigma^\dagger(\mathbf{p})]_- = +p_\mu b_\sigma^\dagger(\mathbf{p}) \quad (72)$$

[17] The Hamiltonian can also be obtained from the second-order Lagrangian presented in ref. 13b, equation (18), by means of the procedure developed by Ostrogradsky[(49)] (see also Weinberg's remark on p. B1325 of ref. 2a). Ostrogradsky's procedure seems not to have been applied in ref. 13 to obtain conjugate momentum operators.

[18] In order not to cloud the essence of the question I assume that transitions $\psi_1 \leftrightarrow \psi_2$ and transitions between states of different signs of energy (as in ref. 41) are irrelevant at the moment. Otherwise, the only correction to be taken into account where necessary is that the commutators (77), (78) should be generalized.[(30)]

Analogous relations exist for the operators $c_\sigma(\mathbf{p})$ and $d_\sigma(\mathbf{p})$. Replacing P_μ by its expansion, this is equivalent to

$$a^\dagger_\sigma(\mathbf{k})[a_\sigma(\mathbf{k}),a_{\sigma'}(\mathbf{p})]_- + [a^\dagger_\sigma(\mathbf{k}),a_{\sigma'}(\mathbf{p})]_- a_\sigma(\mathbf{k}) = -(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{k})a_{\sigma'}(\mathbf{p}) \quad (73)$$

$$b_\sigma(\mathbf{k})[b^\dagger_\sigma(\mathbf{k}),b_{\sigma'}(\mathbf{p})]_- + [b_\sigma(\mathbf{k}),b_{\sigma'}(\mathbf{p})]_- b^\dagger_\sigma(\mathbf{k}) = -(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{k})b_{\sigma'}(\mathbf{p}) \quad (74)$$

$$a^\dagger_\sigma(\mathbf{k})[a_\sigma(\mathbf{k}),a^\dagger_{\sigma'}(\mathbf{p})]_- + [a^\dagger_\sigma(\mathbf{k}),a^\dagger_{\sigma'}(\mathbf{p})]_- a_\sigma(\mathbf{k}) = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{k})a^\dagger_{\sigma'}(\mathbf{p}) \quad (75)$$

$$b_\sigma(\mathbf{k})[b^\dagger_\sigma(\mathbf{k}),b^\dagger_{\sigma'}(\mathbf{p})]_- + [b_\sigma(\mathbf{k}),b^\dagger_{\sigma'}(\mathbf{p})]_- b^\dagger_\sigma(\mathbf{k}) = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{k})b_{\sigma'}(\mathbf{p}) \quad (76)$$

We can list very similar formulas for the states defined by the field function $\psi_2(x)$. Therefore, we deduce the commutation relations

$$[a_\sigma(\mathbf{p}),\ a^\dagger_{\sigma'}(\mathbf{k})]_- = [c_\sigma(\mathbf{p}),\ c^\dagger_{\sigma'}(\mathbf{k})]_- = (2\pi)^3\delta_{\sigma\sigma'}\delta(\mathbf{p}-\mathbf{k}) \quad (77)$$

$$[b_\sigma(\mathbf{p}),\ b^\dagger_{\sigma'}(\mathbf{k})]_- = [d_\sigma(\mathbf{p}),\ d^\dagger_{\sigma'}(\mathbf{k})]_- = (2\pi)^3\delta_{\sigma\sigma'}\delta(\mathbf{p}-\mathbf{k}) \quad (78)$$

It is easy to see that the Hamiltonian is positive-definite and the translational invariance remains in the framework of this description (*cf.* ref. 13). *Note that I did not apply the indefinite metric*, which is a rather obscure concept.

Analogously, from the definitions

$$\mathcal{J}_\mu = -i\sum_i\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\phi_i - \overline{\phi}_i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\phi}_i)}\right\} \quad (79)$$

$$Q = -i\int \mathcal{J}_4(x)\, d^3x \quad (80)$$

and

$$\mathcal{M}_{\mu\nu,\lambda} = x_\mu\mathcal{T}_{\lambda\nu} - x_\nu\mathcal{T}_{\lambda\mu} - i\sum_i\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi_i)}N^{\phi_i}_{\mu\nu}\phi_i + \overline{\phi}_i N^{\overline{\phi}_i}_{\mu\nu}\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\overline{\phi}_i)}\right\} \quad (81)$$

$$M_{\mu\nu} = -i\int \mathcal{M}_{\mu\nu,4}(x)\, d^3x \quad (82)$$

one can find the current operator

$$\mathcal{J}^{(1)}_\mu = i\,[\partial_\alpha\overline{\psi}_1\gamma_{\alpha\mu}\psi_1 - \overline{\psi}_1\gamma_{\mu\alpha}\partial_\alpha\psi_1 + \partial_\alpha\overline{\psi}_2\gamma_{\alpha\mu}\psi_2 - \overline{\psi}_2\gamma_{\mu\alpha}\partial_\alpha\psi_2] \quad (83)$$

and using (81), the spin momentum tensor

$$S^{(1)}_{\mu\nu,\,\lambda} = i\,[\partial_\alpha\overline{\psi}_1\gamma_{\alpha\lambda}N^{\psi_1}_{\mu\nu}\psi_1 + \overline{\psi}_1 N^{\overline{\psi}_1}_{\mu\nu}\gamma_{\lambda\alpha}\partial_\alpha\psi_1 + \partial_\alpha\overline{\psi}_2\gamma_{\alpha\lambda}N^{\psi_2}_{\mu\nu}\psi_2 + \overline{\psi}_2 N^{\overline{\psi}_2}_{\mu\nu}\gamma_{\lambda\alpha}\partial_\alpha\psi_2] \quad (84)$$

If the Lorentz group generators (the $j = 1$ case) are defined from

$$\overline{\Lambda}\gamma_{\mu\nu}\Lambda a_{\mu\alpha}a_{\nu\beta} = \gamma_{\alpha\beta} \tag{85}$$

$$\overline{\Lambda}\Lambda = 1 \tag{86}$$

$$\overline{\Lambda} = \gamma_{44}\Lambda^{\dagger}\gamma_{44} \tag{87}$$

then in order to keep the Lorentz covariance of the Weinberg equations and of the Lagrangian (45), one should use the following generators:

$$N_{\mu\nu}^{\psi_1,\psi_2(j=1)} = -N_{\mu\nu}^{\overline{\psi}_1,\overline{\psi}_2(j=1)} = \frac{1}{6}\gamma_{5,\mu\nu} \tag{88}$$

The matrix $\gamma_{5,\mu\nu} = i[\gamma_{\mu\lambda}, \gamma_{\nu\lambda}]_-$ is defined to be Hermitian. The choice of generators for Lorentz transformations has also been regarded in ref. 16. Due to the fact that the set of Weinberg states is degenerate, one can also consider the situation when one Weinberg equation [*e.g.*, (21)] transforms into another [*e.g.*, (24)]. This case corresponds to the possibility of combining pure Lorentz transformations with transformations of the inversion group; the corresponding rules are different from (85)–(87).

The quantized charge operator and the quantized spin operator follow immediately from (83) and (84):

$$Q^{(1)} = \sum_{\sigma}\int\frac{d^3\mathbf{p}}{(2\pi)^3}[a_\sigma^\dagger(\mathbf{p})a_\sigma(\mathbf{p}) - b_\sigma(\mathbf{p})b_\sigma^\dagger(\mathbf{p}) + c_\sigma^\dagger(\mathbf{p})c_\sigma(\mathbf{p}) - d_\sigma(\mathbf{p})\,d_\sigma^\dagger(\mathbf{p})] \tag{89}$$

$$(W^{(1)}\cdot n)/m = \sum_{\sigma\sigma'}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{m^2E_p}\overline{u}_1^{\sigma}(\mathbf{p})(E_p\gamma_{44} - i\gamma_{4i}p_i)\,I\otimes(\mathbf{J}\mathbf{n})u_1^{\sigma'}(\mathbf{p}) \times[a_\sigma^\dagger(\mathbf{p})a_{\sigma'}(\mathbf{p}) + c_\sigma^\dagger(\mathbf{p})c_{\sigma'}(\mathbf{p}) - b_\sigma(\mathbf{p})b_{\sigma'}^\dagger(\mathbf{p}) - d_\sigma(\mathbf{p})d_{\sigma'}^\dagger(\mathbf{p})] \tag{90}$$

(provided that the frame is chosen in such a way that $\mathbf{n} \parallel \mathbf{p}$ is along the third axis). It is easy to verify that the eigenvalues of the charge operator are ± 1, and of the Pauli-Lyuban'sky spin operator are

$$\xi_\sigma^*(\mathbf{J}\mathbf{n})\xi_{\sigma'} = +1,\ 0,\ -1 \tag{91}$$

in the massive case and ± 1 in the massless case.[19] Now we can answer why

[19] See the discussion of the massless limit of the Weinberg bispinors in refs. 35 and 51. While in the massless limit $W_\mu n_\mu = 0$, this does not signify that W_μ would always be equal to zero; in this case we already cannot define a normalized spacelike vector n_μ whose space part is parallel to the vector $\mathbf{p}$. It becomes lightlike.

"a queer reduction of degrees of freedom" happened in refs. 22, 23, 25. The origin of this surprising fact follows from Hayashi (ref. 22, p. 498): The requirement "*that the physical realizable state satisfies a quantal version of the generalized Lorentz condition*," formulas (18) of ref. 22,[20] permits one to eliminate the upper (or down) part of the Weinberg "bispinor" and to remove transverse components of the remaining part by means of the "gauge" transformation (44), which "*ensures the massless skew-symmetric field is longitudinal* ." The reader can be convinced of this "obvious fact" by looking at the explicit form of the Pauli–Lyuban'sky operator (90). Taking into account both positive- and negative-energy solutions (*cf.* ref. 25) in the Lagrangian (40) and *not* applying the generalized Lorentz condition (*cf.* refs. 22, 23), we are able to account for both transverse and longitudinal components, *i.e.*, to describe a $j = 1$ particle. Furthermore, one can say even more simply that the application of the generalized Lorentz condition may be successful to the nonzero-energy states of helicities ± 1,[21] so in earlier works, as a matter of fact, the authors implied the existence of such states. On the other hand, longitudinal components of the Weinberg fields are directly linked with the mass of a $j = 1$ particle,[(51)] and, possibly, with the concept of the $\mathbf{B}^{(3)}$ Evans–Vigier field.[(39)] This fact can provide a deeper understanding of relations between Casimir invariants of a particle field and space-time structures. The present wisdom does *not* contradict the Weinberg theorem nor the classical limit, (27), (28) of the previous section. Thanks to the mapping between the antisymmetric tensor and Weinberg formulations, the conclusion is valid for both the Weinberg $2(2j + 1)$-component "bispinor" and the antisymmetric (skew-symmetric) tensor field. Thus, we have now proven that a photon (a $j = 1$ massless particle) can possess spin degrees of freedom, in accordance with experiment. The contradictory claims about the pure "longitudinal nature" of quantized antisymmetric fields which have been made since the sixties are unreasonable. We can suggest an analogy considering the modified electrodynamics recently proposed by Evans and Vigier. In fact, the authors of the earlier "longitudinal" papers "align themselves" with the concept of the $\mathbf{B}^{(3)}$ field (named the Kalb–Ramond field), but, surprisingly, they reject transverse modes (after quantization)!? By the way, it is obvious from the

[20] Read: "a quantal version" of the Maxwell equations imposed on the state vectors in the Fock space. Applying them leads to the case when (90) is equal to zero *identically*. Nonetheless, such a procedure should be taken cautiously; see, *e.g.*, ref. 13, Table 2, for a discussion of the acausal physical dispersion of equations (4.19) and (4.20) of ref. 2b, "which *are just Maxwell's free-space equations for left- and right-circularly polarized radiation*." See also footnote 1 in ref. 28c. The existence of 'acausal' solutions is probably connected with the indefinite metric problem, with the appearance of ghost states in the gauge models, and with the concept of 'action-at-a-distance.'[(67)]

[21] If the energy is equal to zero, then in my opinion there is no sense in speaking about helicity at all.

consideration of the similar construct in the $(1/2, 0) \oplus (0, 1/2)$ representation that on an equal footing those authors could claim that a $j = 1/2$ massless neutrino field would be pure longitudinal, too. Simply speaking, such claims are absurd.

Finally, for the sake of completeness let me rewrite the Lagrangians presented above in the 12-component form:

$$\mathcal{L}^{(1)} = -\partial_\mu \overline{\Psi} \Gamma_{\mu\nu} \partial_\nu \Psi - m^2 \overline{\Psi} \Psi \tag{92}$$

where

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad \overline{\Psi} = (\psi_1^\dagger \ \psi_2^\dagger) \cdot \begin{pmatrix} \gamma_{44} & 0 \\ 0 & -\gamma_{44} \end{pmatrix} \tag{93}$$

are the doublet wave functions,

$$\Gamma_{\mu\nu} = \begin{pmatrix} \gamma_{\mu\nu} & 0 \\ 0 & -\gamma_{\mu\nu} \end{pmatrix}, \qquad \Gamma^5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \Gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{94}$$

The Lagrangian $\mathcal{L}^{(2)}$ can be written in a similar fashion:

$$\begin{aligned} \mathcal{L}^{(2)} = &-\partial_\mu^\dagger \Psi^{(1)\dagger} \Gamma_{\mu\nu} \Gamma^5 \Gamma^0 \partial_\nu^\dagger \psi^{(2)} - \partial_\mu \Psi^{(2)\dagger} \Gamma_{\mu\nu} \Gamma^5 \Gamma^0 \partial_\nu \Psi^{(1)} \\ &- m^2 \Psi^{(1)\dagger} \Gamma^5 \Gamma^0 \Psi^{(2)} + m^2 \Psi^{(2)\dagger} \Gamma^5 \Gamma^0 \Psi^{(1)} \end{aligned} \tag{95}$$

One can conclude this section: the generalized Lorentz condition can be incompatible with the specific properties of the antisymmetric tensor field deduced from the ordinary approach of classical physics, i.e., its application can lead (and did lead in earlier papers) to the loss of information about either transverse or longitudinal modes of the antisymmetric tensor field. The connection of the present model with the Bargmann–Wightman–Wigner-type quantum field theories deserves further elaboration. As a matter of fact, the present model develops Weinberg's and Ahluwalia's ideas of the Dirac-like description of bosons on an equal footing with fermions, *i.e.*, on the ground of the $(j, 0) \oplus (0, j)$ representation of the Lorentz group.

4. WEINBERG PROPAGATORS

According to the Feynman–Dyson–Stueckelberg ideas, a causal propagator has to be constructed by using the formula (e.g., ref. 41, p. 91).

$$\begin{aligned} S_F(x_2, x_1) = \int \frac{d^3\mathbf{k}\ m}{(2\pi)^3 E_k} [&\theta(t_2 - t_1)\ a\ u^\sigma(k) \otimes \overline{u}^\sigma(k) e^{-ikx} \\ &+ \theta(t_1 - t_2)\ b\ v^\sigma(k) \otimes \overline{v}^\sigma(k) e^{ikx}] \end{aligned} \tag{96}$$

($x = x_2 - x_1$). In the $j = 1/2$ Dirac theory we obtain

$$S_F(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{\hat{k} + m}{k^2 - m^2 + i\varepsilon} \tag{97}$$

provided that the constants a and b are determined by imposing

$$(i\hat{\partial}_2 - m) S_F(x_2, x_1) = \delta^{(4)}(x_2 - x_1) \tag{98}$$

namely $a = -b = 1/i$.

However, in the framework of the Weinberg theory,(2) which is a generalization of the Dirac ideas to higher spins, attempts at constructing a covariant propagator in such a way have failed. For example, on the p. B1324 of ref. 2a Weinberg writes: "Unfortunately, the propagator arising from Wick's theorem is *not* equal to the covariant propagator except for $j = 0$ and $j = 1/2$. The trouble is that the derivatives act on the $\varepsilon(x) = \theta(x) - \theta(-x)$ in $\Delta^C(x)$ as well as on the functions[22] Δ and Δ_1. This gives rise to extra terms proportional to equal-time δ functions and their derivatives . . . The cure is well known: . . . compute the vertex factors using only the original covariant part of [the Hamiltonian] $\mathcal{H}(x)$; do not use [the Wick propagator] for internal lines; instead use the covariant propagator [the formula (5.8) in ref. 2a]." The propagator recently proposed in refs. 35c and 35d (see also other papers of the same author) is the causal propagator: "Only the physically acceptable causal solutions of the Weinberg equations enter these propagators." However, this does not satisfy us fundamentally since the old problem remains: the Feynman–Dyson propagator is not the Green's function of the Weinberg equation. The covariant propagator presented in ref. 5, while a Green's function of the (1, 0) ⊕ (0, 1) equation, would propagate kinematically spurious solution.(35c) Our aim in the following work is to consider the problem of constructing propagators in the framework of the model proposed in the previous sections.

A set of four equations has been proposed in section 2. We consider the most general case. Let us check if the sum of four equations ($x = x_2 - x_1$)

$$[\gamma_{\mu\nu}\partial_\mu\partial_\nu - m^2] \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} [\theta(t_2 - t_1)\, a\, \mathcal{U}_1^{\sigma(1)}(\mathbf{p}) \otimes \overline{\mathcal{U}}_1^{\sigma(1)}(\mathbf{p}) e^{ipx}$$
$$+ \theta(t_1 - t_2)\, b\, \mathcal{V}_1^{\sigma(1)}(\mathbf{p}) \otimes \overline{\mathcal{V}}_1^{\sigma(1)}(\mathbf{p}) e^{-ipx}]$$
$$+ [\gamma_{\mu\nu}\partial_\mu\partial_\nu + m^2] \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} [\theta(t_2 - t_1)\, c\, \mathcal{U}_2^{\sigma(1)}(\mathbf{p}) \otimes \overline{\mathcal{U}}_2^{\sigma(1)}(\mathbf{p}) e^{ipx}$$

[22] In the cited paper the following notation has been used: $\Delta_1(x) \equiv i[\Delta_+(x) + \Delta_+(-x)]$, $\Delta(x) \equiv \Delta_+(x) - \Delta_+(-x)$, and $i\Delta_+(x) \equiv (2\pi)^{-3} \int (d^3\mathbf{p}/2E_p) \exp(ipx)$.

$$+ \theta(t_1 - t_2)\, d\, \mathcal{V}_2^{\sigma(1)}(\mathbf{p}) \otimes \overline{\mathcal{V}}_2^{\sigma(1)}(\mathbf{p}) e^{-ipx}]$$

$$+ [\tilde{\gamma}_{\mu\nu}\partial_\mu\partial_\nu - m^2] \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} [\theta(t_2 - t_1)\, g\, \mathcal{U}_2^{\sigma(2)}(\mathbf{p}) \otimes \overline{\mathcal{U}}_2^{\sigma(2)}(\mathbf{p}) e^{ipx}$$

$$+ \theta(t_1 - t_2)\, h\, \mathcal{V}_2^{\sigma(2)}(\mathbf{p}) \otimes \overline{\mathcal{V}}_2^{\sigma(2)}(\mathbf{p}) e^{-ipx}] = \delta^{(4)}(x_2 - x_1) \tag{99}$$

can be satisfied by a definite choice of the constants *a, b, etc*. In the process of calculation I assume that the set of the analogs of the "Pauli spinors" in the (1, 0) or (0, 1) space is complete and is normalized to $\delta_{\sigma\sigma'}$.

Simple calculations yield

$$\partial_\mu^{x_2}\partial_\nu^{x_2}[a\theta(t_2-t_1)e^{ip(x_2-x_1)} + b\theta(t_1-t_2)e^{-ip(x_2-x_1)}]$$

$$= -[ap_\mu p_\nu\theta(t_2-t_1)\exp[ip(x_2-x_1)] + bp_\mu p_\nu\theta(t_1-t_2)\exp[-ip(x_2-x_1)]]$$

$$+ a[-\delta_{\mu 4}\delta_{\nu 4}\delta'(t_2-t_1) + i(p_\mu\delta_{\nu 4} + p_\nu\delta_{\mu 4})\delta(t_2-t_1)]\exp[i\mathbf{p}(\mathbf{x}_2-\mathbf{x}_1)]$$

$$+ b[\delta_{\mu 4}\delta_{\nu 4}\delta'(t_2-t_1) + i(p_\mu\delta_{\nu 4} + p_\nu\delta_{\mu 4})\delta(t_2-t_1)]\exp[-i\mathbf{p}(\mathbf{x}_2-\mathbf{x}_1)] \tag{100}$$

and

$$\mathcal{U}_1^{(1)}\overline{\mathcal{U}}_1^{(1)} = \frac{1}{2}\begin{pmatrix} m^2\mathbb{1} & S_p\otimes S_p \\ \overline{S}_p\otimes\overline{S}_p & m^2\mathbb{1}\end{pmatrix}, \quad \mathcal{U}_2^{(1)}\overline{\mathcal{U}}_2^{(1)} = \frac{1}{2}\begin{pmatrix} -m^2\mathbb{1} & S_p\otimes S_p \\ \overline{S}_p\otimes\overline{S}_p & -m^2\mathbb{1}\end{pmatrix} \tag{101}$$

$$\mathcal{U}_1^{(2)}\overline{\mathcal{U}}_1^{(2)} = \frac{1}{2}\begin{pmatrix} -m^2\mathbb{1} & \overline{S}_p\otimes\overline{S}_p \\ S_p\otimes S_p & -m^2\mathbb{1}\end{pmatrix}, \quad \mathcal{U}_2^{(2)}\overline{\mathcal{U}}_2^{(2)} = \frac{1}{2}\begin{pmatrix} m^2\mathbb{1} & \overline{S}_p\otimes\overline{S}_p \\ S_p\otimes S_p & m^2\mathbb{1}\end{pmatrix} \tag{102}$$

where

$$S_p = m + (\mathbf{Jp}) + \frac{(\mathbf{Jp})^2}{E+m} \tag{103}$$

$$\overline{S}_p = m - (\mathbf{Jp}) + \frac{(\mathbf{Jp})^2}{E+m} \tag{104}$$

Due to the fact that

$$[E - (\mathbf{Jp})]\, S_p \otimes S_p = m^2\,[E + (\mathbf{Jp})] \tag{105}$$

$$[E + (\mathbf{Jp})]\, \overline{S}_p \otimes \overline{S}_p = m^2\,[E + (\mathbf{Jp})] \tag{106}$$

after simplifying the left side of (99) and comparing it with the right side, we find that the causal propagator is admitted by using the Wick formula for the time-ordered particle operators, provided that the constants are equal to $1/4im^2$. It is necessary to consider all four equations (21), (22), (24), and (25).

The $j = 1$ analogs of the formula (97) for the Weinberg propagators follow from the formula (3.6) of ref. 35d immediately:

$$S_F^{(1)}(p) = -\frac{1}{i(2\pi)^4(p^2 + m^2 - i\varepsilon)}[\gamma_{\mu\nu}p_\mu p_\nu - m^2] \qquad (107)$$

$$S_F^{(2)}(p) = -\frac{1}{i(2\pi)^4(p^2 + m^2 - i\varepsilon)}[\gamma_{\mu\nu}p_\mu p_\nu + m^2] \qquad (108)$$

$$S_F^{(3)}(p) = -\frac{1}{i(2\pi)^4(p^2 + m^2 - i\varepsilon)}[\tilde{\gamma}_{\mu\nu}p_\mu p_\nu + m^2] \qquad (109)$$

$$S_F^{(4)}(p) = -\frac{1}{i(2\pi)^4(p^2 + m^2 - i\varepsilon)}[\tilde{\gamma}_{\mu\nu}p_\mu p_\nu - m^2] \qquad (110)$$

The conclusions are that one can construct an analog of the Feynman–Dyson propagator for the $2(2j + 1)$ model and, hence, a "local" theory, provided that the Weinberg states are "quadrupled" in the $j = 1$ case. They cannot propagate separately from each other (compare with the Dirac $j = 1/2$ case).

5. MASSLESS LIMIT: CAN THE SIX-COMPONENT WEINBERG–TUCKER–HAMMER EQUATIONS DESCRIBE THE ELECTROMAGNETIC FIELD?

In previous sections the equivalence of the Weinberg–Tucker–Hammer approach and the Proca approach for describing $j = 1$ states has been found. The $2(2j + 1)$-component wave functions are given by (39) and by the formulas obtained after applying inversion group operations to (39). The aim of the present section is to consider the conditions under which the Weinberg–Tucker–Hammer $j = 1$ equations can be transformed to (4.21) and (4.22) of ref. 2b:

$$\nabla \times [\mathbf{E} - i\mathbf{B}] + i(\partial/\partial t)[\mathbf{E} - i\mathbf{B}] = 0 \qquad \text{[ref. 2b, (4.21)]}$$

$$\nabla \times [\mathbf{E} + i\mathbf{B}] - i(\partial/\partial t)[\mathbf{E} + i\mathbf{B}] = 0 \qquad \text{[ref. 2b, (4.22)]}$$

By using the bivector interpretation of ψ (in the chiral representation) and the explicit forms of the Barut–Muzinich–Williams matrices, we are able to recast the $j = 1$ Tucker–Hammer equation (13), which is free of tachyonic solutions, or the Proca equation (8) of Section 2, to the form

$$m^2 E_i = -\frac{\partial^2 E_i}{\partial t^2} + \varepsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial B_k}{\partial t} + \frac{\partial}{\partial x_i}\frac{\partial E_j}{\partial x_j} \tag{111}$$

$$m^2 B_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial E_k}{\partial t} + \frac{\partial^2 B_i}{\partial x_j^2} - \frac{\partial}{\partial x_i}\frac{\partial B_j}{\partial x_j} \tag{112}$$

The Klein–Gordon equation (the D'Alembert equation in the massless limit)

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_i^2}\right) F_{\mu\nu} = -m^2 F_{\mu\nu} \tag{113}$$

is implied ($c = \hbar = 1$). Introducing vector operators, we write equations in the following form:

$$\frac{\partial}{\partial t}\,\text{curl}\,\mathbf{B} + \text{grad div}\,\mathbf{E} - \frac{\partial^2 \mathbf{E}}{\partial t^2} = m^2 \mathbf{E} \tag{114}$$

$$\nabla^2 \mathbf{B} - \text{grad div}\,\mathbf{B} + \frac{\partial}{\partial t}\,\text{curl}\,\mathbf{E} = m^2 \mathbf{B} \tag{115}$$

Taking into account the definitions

$$\rho_e = \text{div}\,\mathbf{E}, \qquad \mathbf{J}_e = \text{curl}\,\mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} \tag{116}$$

$$\rho_m = \text{div}\,\mathbf{B}, \qquad \mathbf{J}_m = -\frac{\partial \mathbf{B}}{\partial t} - \text{curl}\,\mathbf{E} \tag{117}$$

relations of the vector algebra (**X** is an arbitrary vector):

$$\text{curl curl}\,\mathbf{X} = \text{grad div}\,\mathbf{X} - \nabla^2 \mathbf{X} \tag{118}$$

and the Klein–Gordon equation (113), we obtain two equivalent sets of equations which complete the Maxwell set. The first is

$$\frac{\partial \mathbf{J}_e}{\partial t} + \text{grad}\,\rho_e = m^2 \mathbf{E} \tag{119}$$

$$\frac{\partial \mathbf{J}_m}{\partial t} + \text{grad}\,\rho_m = 0 \tag{120}$$

and the second is

$$\text{curl}\mathbf{J}_m = 0 \tag{121}$$

$$\text{curl}\mathbf{J}_e = -m^2 \mathbf{B} \tag{122}$$

One can obtain the equations in different systems of units after one recalls, *e.g.*, the relations of the Appendix of ref. 52. Also recall that the Weinberg set of equations [and, hence, equations (119)–(122)[23]] can be obtained on the basis of a very few postulates, in fact, by using the Lorentz transformation rules for the Weinberg bivector (or for the antisymmetric tensor field) and the Ryder–Burgard relation.(13,14,17–19)

In the massless case the situation is different. First, the set of equations (117) with the left side chosen to be zero is "an identity satisfied by certain space-time derivatives of $F_{\mu\nu}$ . . . , namely,(53–55)

$$\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} + \frac{\partial F_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial F_{\sigma\mu}}{\partial x^{\nu}} = 0\text{''} \tag{127}$$

I believe that a similar consideration for the dual field $\tilde{F}_{\mu\nu}$ as in refs. 53 and 54 can reveal that the same is true for the first equations (116). So, in the massless case we come to the problem of the interpretation of the charge and currents.

Second, in order to satisfy the massless equations (121), (122) one should assume that the currents are represented in the gradient forms of some scalar fields $\chi_{e,m}$. What physical significance do these chi functions have? In the massless case the charge densities are [see equations (119), (120)]

$$\rho_e = -\frac{\partial \chi_e}{\partial t} + \text{const}, \qquad \rho_m = -\frac{\partial \chi_m}{\partial t} + \text{const} \tag{128}$$

which tells us that ρ_e and ρ_m are constants, provided that the primary functions $\chi_{e,m}$ are linear functions in time (decreasing or increasing?). It is useful to compare the resulting equations for $\rho_{e,m}$ and $\mathbf{J}_{e,m}$ and the appearence of the functions $\chi_{e,m}$ with the 5-potential formulation of electromagnetic theory(54) (see also refs. 24, 55–59). I believe this concept can also be useful for explanation of the $E = 0$ solutions in higher-spin equations(60,61,35) which have been "baptized" by Moshinsky and Del Sol(62) as " 'relativistic cockroach nest.' Next, I would like to note the following. We can obtain Maxwell's

[23] Beginning with the dual massive equations and setting $\mathbf{C} \equiv \mathbf{E}$, $\mathbf{D} \equiv \mathbf{B}$, we obtain

$$\frac{\partial \mathbf{J}_e}{\partial t} + \text{grad } \rho_e = 0 \tag{123}$$

$$\frac{\partial \mathbf{J}_m}{\partial t} + \text{grad } \rho_m = m^2 \mathbf{B} \tag{124}$$

and

$$\text{curl } \mathbf{J}_e = 0 \tag{125}$$

$$\text{curl } \mathbf{J}_m = m^2 \mathbf{E} \tag{126}$$

This would signify that the physical content spanned by massive dual fields would be different. The reader can easily find parity-conjugated equations from (24) and (25).

free-space equations for a definite choice of the χ_e and χ_m, namely, in the case that they are constants. In ref. 56 it was mentioned that solutions of (4.21), (4.22) of ref. 2b satisfy equations of the type (111), (112), "*but not always vice versa.*" Interpretation of this statement and investigations of (13) with different initial and boundary conditions (or of the functions χ) deserve further elaboration (both theoretical and experimental).

The question also arises of the transformation of the field function (39) from one frame to another. I would like to draw attention to the remarkable fact which follows from a consideration of the problem in the momentum representation. At first sight, one might conclude that under a transfer from one frame to another one has to describe the field by the Lorentz-transformed function $\psi'(\mathbf{p}) = \Lambda(\mathbf{p})\psi(\mathbf{p})$. However, if we take into account the possibility of combining the Lorentz, dual (chiral), and parity transformations in the case of higher spin equations[24] and that all the equations for the four functions (21), (22), (24), and (25) reduce to the equations for **E** and **B**, which appear to be the same in the massless limit, one could come to a different situation. The four bispinors $\mathcal{U}_1^{\sigma(1)}(\mathbf{p})$, $\mathcal{U}_2^{\sigma(1)}(\mathbf{p})$, $\mathcal{U}_1^{\sigma(2)}(\mathbf{p})$, and $\mathcal{U}_2^{\sigma(2)}(\mathbf{p})$ [see (52), (53), (57), and (58)] form a complete set [as well as the transformed ones $\Lambda(\mathbf{p})\mathcal{U}_i^{\sigma(k)}(\mathbf{p})$] for each value of σ. Namely,

$$a_1\mathcal{U}_1^{\sigma(1)}(\mathbf{p})\overline{\mathcal{U}}_1^{\sigma(1)}(\mathbf{p}) + a_2\mathcal{U}_2^{\sigma(1)}(\mathbf{p})\overline{\mathcal{U}}_2^{\sigma(1)}(\mathbf{p}) \\ + a_3\mathcal{U}_1^{\sigma(2)}(\mathbf{p})\overline{\mathcal{U}}_1^{\sigma(2)}(\mathbf{p})\ a_4\mathcal{U}_2^{\sigma(2)}(\mathbf{p})\overline{\mathcal{U}}_2^{\sigma(2)}(\mathbf{p}) = \mathbb{1} \tag{129}$$

The constants a_i are defined by the choice of the normalization of the bispinors. In any other frame we are able to obtain the primary wave function by choosing appropriate coefficients c_i^k of the expansion of the wave function (in fact, using appropriate dual rotations and inversions)

$$\Psi(\mathbf{p}) = \sum_{i,k=1,2} c_i^k\mathcal{U}_i^{(k)}(\mathbf{p}) \tag{130}$$

The same statement should be valid for negative-energy solutions, since their explicit forms coincide with those of positive-energy bispinors in the case of the Hammer–Tucker formulation for a $j = 1$ boson.[(5)] Using the plane-wave expansion, one can prove this conclusion in the coordinate representation. Thus, the question of what we observe in experiment would be solved depending on the fixing of the relative phase factor between left and right parts of the field function (indeed, between **E** and **B**) by the appropriate physical conditions in which we are interested.

Finally, I note that the massless case reveals a very strange thing.[25] The massless equations (121), (122) written in the integral form lead to the

[24] This possibility was discovered earlier and investigated in ref. 13.

[25] I am grateful to Dr. A. E. Chubykalo for pointing out this fact and for discussions.

conclusion that $\oint \mathbf{J}_{e,m} \cdot d\mathbf{l} = 0$. This is obviously unacceptable from the viewpoint of experiment. Thus, we have to conclude that either the $j = 1$ field cannot be massless or there exist hidden parameters on which all field functions (and, probably, space-time characteristics) depend.

Finally, let me mention that in the nonrelativistic limit $c \to \infty$ one obtains the dual Levi-Leblond "Galilean electrodynamics."[63,64]

The main conclusion of the paper is as follows[26]: The Weinberg–Tucker–Hammer massless equations (or the Proca equations for $F_{\mu\nu}$) [see also (111) and (112)] are equivalent to the Maxwell equations for a definite choice of initial and boundary conditions, which proves their consistency. Their massless limit was shown in ref. 35 to be free of kinematic acausalities, as opposed to (4.21) and (4.22) of ref. 2b. The Weinberg–Tucker–Hammer approach permits us to clarify the question of the claimed 'longitudinal nature' of the antisymmetric tensor field. It is free of the problem of the indefinite metric in the Fock space. The $j = 1$ bosons are considered in a very similar fashion to fermions in the Dirac approach. This provides a convenient mathematical formalism for discussing properties of the $j = 1$ bosons with respect to discrete symmetry operations. Therefore, we have to agree with Weinberg, who wrote in connection with equations (4.21) and (4.22), "*The fact that these field equations are of first order for any spin seems to me to be of no great significance*" (ref. 2b, p. B888). In the meantime, I would not like to denigrate theories based on the use of the vector potentials, *i.e.*, of the $D(1/2, 1/2)$ representation of the Lorentz group. While the description of the $j = 1$ massless field using this representation contradicts the Weinberg theorem $B - A = \lambda$, which signifies that we do not have well-defined creation and annihilation operators in the beginning of a quantization procedure, one cannot forget the significant achievements of these theories. The formalism proposed here could be helpful only if we need to go beyond the framework of the Standard Model, *i.e.*, if we find reliable experimental results which cannot have a satisfactory explanation on the basis of the concept of a minimal coupling introduced in the conventional manner (see, *e.g.*, ref. 14 for a discussion of the neutrino model, which forbids such a form of the interaction).

Many questions related to the problem of longitudinal modes of the $j = 1$ field, their relations with tachyonic models (particularly with the concept of action at a distance and Recami's extended relativity), with the problem of the interpretations of mass and spin, and with the problem of gauge degrees of freedom remain for future research.

[26] This conclusion also follows from the results of refs. 13, 35, 28–32, 39, and 40 and ref. 2b provided that the fact that $(\mathbf{Jp})$ has no inverse matrix has been taken into account.

ACKNOWLEDGMENTS

I am glad to express my gratitude to Prof. D. V. Ahluwalia for useful advice which enabled me to begin work in this direction. His answers to my questions and firm support contributed greatly to the ideas presented here and in my previous publications.

A considerable part of this paper was written in August 1994, and I have benefited from discussion of the ideas presented above in many symposiums and in private correspondence. Recently I learnt about relevant papers of other authors. Work of H. A. Múnera and O. Guzmán on the longitudinal solutions of relativistic wave equations, of A. E. Chubykalo and R. Smirnov-Rueda[(67)] on the 'action-at-a-distance' concept in classical electrodynamics (*cf.* the research of Belinfante in QED[(68)]), and of E. Recami, W. Rodrigues, and J. Vaz on superluminal phenomena, which have recently been observed, also deserve attention and verification. Furthermore, as I also learnt recently, some of the issues considered by me in the present paper were considered in older work.[(65,66)] My papers, though connected with previous considerations, are in my opinion, more relevant to modern field theory. The group-theoretic basis for my research has been proposed in the papers of E. Wigner, Y. S. Kim, and S. Weinberg.

I greatly appreciate valuable discussions with Profs. R. N. Faustov, I. G. Kaplan, Y. S. Kim, A. Mondragon, M. Moreno, M. Moshinsky, E. Recami, S. Roy, M. Sachs, N. B. Skachkov, Yu. F. Smirnov, J. Vaz, and G. Ziino.

I am grateful to Zacatecas University, México, for a full professorship. This work has been partly supported by the Mexican Sistema National de Investigadores, the Programa de Apoyo a la Carrera Docente, and by CONACyT, Mexico, under research project 0270P-E.

REFERENCES

1. H. Joos, *Fortschr. Phys.* **10** (1962) 65.
2. (a) S. Weinberg, *Phys. Rev. B* **133** (1964) 1318; (b) **134** (1964) 882; (c) **181** (1969) 1893.
3. D. L. Weaver, C. L. Hammer, and R. H. Good, Jr., *Phys. Rev. B* **135** (1964) 241.
4. Yu. V. Novozhilov, *Vvedenie v teoriyu elementarnykh chastitz*, Moscow, Nauka (1971) [English translation: *Introduction to Elementary Particle Theory*, Pergamon Press, Oxford (1975)].
5. R. H. Tucker and C. L. Hammer, *Phys. Rev. D* **3** (1971) 2448.
6. C. L. Hammer, S. C. McDonald, and D. L. Pursey, *Phys. Rev.* **171** (1968) 1349; D. Shay and R. H. Good, Jr., *Phys. Rev.* **179** (1969) 1410.
7. P. A. Vlasov, *Ukrain. Fiz. Zhurn.* **18** (1973) 1238; **22** (1977) 951 [in Russian].
8. H. M. Ruck and W. Greiner, *J. Phys. A* **3** (1977) 657.
9. F. D. Santos, *Phys. Lett. B* **175** (1986) 110; F. D. Santos and H. Van Dam, *Phys. Rev. C* **34** (1991) 250.
10. L. Lukaszuk and L. Szimanowski, *Phys. Rev. D* **36** (1987) 2440.

11. (a) V. V. Dvoeglazov and N. B. Skachkov, JINR Communications P2-84-199, JINR, Dubna, 1984; (b) P2-87-882, JINR, Dubna, 1987; (c) *Yadern. Fiz.* **48** (1988) 1770 [English translation: *Sov. J. Nucl. Phys.* **48** (1988) 1065].
12. V. K. Mishra *et al., Phys. Rev. C* **43** (1991) 801.
13. (a) D. V. Ahluwalia, M. B. Johnson, and T. Goldman, *Phys. Lett. B* **316** (1993) 102; (b) D. V. Ahluwalia and T. Goldman, *Mod. Phys. Lett. A* **8** (1993) 2623.
14. D. V. Ahluwalia, *Int. J. Mod. Phys.* **11** (1996) 1855.
15. E. P. Wigner, In *Group Theoretical Concepts and Methods in Elementary Particle Physics*, F. Gürsey, ed., Gordon and Breach, New York (1965).
16. (a) A. Sankaranarayanan and R. H. Good, Jr., *Nuovo Cim.* **36** (1965) 1303; (b) *Phys. Rev.* **140** (1965) B509; (c) A. Sankaranarayanan, *Nuovo Cim.* **38** (1965) 889.
17. V. V. Dvoeglazov, *Rev. Mex. Fis. Suppl.* **41** (1995) 159; *Int. J. Theor. Phys.* **34** (1995) 2467.
18. V. V. Dvoeglazov, *Nuovo Cim.* **111B** (1996) 483.
19. V. V. Dvoeglazov, *Hadronic J. Suppl.* **10** (1995) 349.
20. F. Chang and F. Gürsey, *Nuovo Cim. A* **63** (1969) 617.
21. Y. Takahashi and R. Palmer, *Phys. Rev. D* **1** (1970) 2974.
22. K. Hayashi, *Phys. Lett.* **44B** (1973) 497.
23. M. Kalb and P. Ramond, *Phys. Rev. D* **9** (1974) 2273.
24. O. M. Boyarkin, *ZhETF* **75** (1978) 26; *Izv. VUZov:fiz.* **24**(11) (1981) 29 [English translation: *Sov. Phys. J.* **24** (1981) 1003]; *Vestzi AN BSSR* **5** (1983) 94.
25. L. V. Avdeev and M. V. Chizhov, *Phys. Lett. B* **321** (1994) 212; A queer reduction of degrees of freedom, Preprint JINR E2-94-263 (hep-th/9407067), Dubna, 1994.
26. D. Birmingham *et al., Phys. Rep.* **209** (1991) 129.
27. M. V. Chizhov, *Mod. Phys. Lett. A* **8** (1993) 2753; New tensor interactions and the $K_L - K_S$ mass difference, Preprint JINR E2-94-253 (hep-ph/9407237), JINR, Dubna, 1994; New tensor interactions in μ decay, Preprint JINR E2-94-254 (hep-ph/9407236), JINR, Dubna, 1994.
28. (a) V. V. Dvoeglazov, *Hadronic J.* **16** (1993) 459; (b) *Rev. Mex. Fis. Suppl.* **40** (1994) 352; (c) V. V. Dvoeglazov, Yu. N. Tyukhtyaev, and S. V. Khudyakov, *Izv. VUZov:fiz.* **37**(9) (1994) 110 [English translation: *Russ. Phys. J.* **37** (1994) 898].
29. V. V. Dvoeglazov, *Helv. Phys. Acta* **70** (1997) 677.
30. V. V. Dvoeglazov, *Helv. Phys. Acta* **70** (1997) 686.
31. V. V. Dvoeglazov, *Helv. Phys. Acta* **70** (1997) 697.
32. V. V. Dvoeglazov, Can the Weinberg–Tucker–Hammer equations describe the electromagnetic field? Preprint EFUAZ FT-94-09-REV (hep-th/9410174), Zacatecas, Oct. 1994.
33. D. Lurie, *Particles and Fields*, Interscience, New York (1968).
34. paA. O. Barut, I. Muzinich, and D. Williams, *Phys. Rev.* **130** (1963) 442.
35. (a) D. V. Ahluwalia and D. J. Ernst, *Phys. Lett. B* **287** (1992) 18; (b) *Mod. Phys. Lett. A* **7** (1992) 1967; (c) *Phys.* Rev. C **45** (1992) 3010; (d) *Int. J. Mod. Phys. E* **2** (1993) 397.
36. P. A. M. Dirac, *Proc. Roy. Soc. A* **117** (1928) 610; **118** (1928) 351.
37. V. I. Starzhev and S. I. Kruglov, *Acta Phys. Polon. B* **8** (1977) 807; V. I. Strazhev, *Acta Phys. Polon. B* **9** (1978) 449; *Int. J. Theor. Phys.* **16** (1977) 111.
38. V. B. Berestetskiı, E. M. Lifshits, and L. P. Pitaevskiı, *Relativistic Quantum Theory. Vol. I*, Pergamon Press, Oxford (1979).
39. M. W. Evans, *Physica A* **214** (1995) 605.
40. V. V. Dvoeglazov, About the question of 'longitudinal' nature of the antisymmetric tensor field after quantization, Preprint EFUAZ FT-95-16-REV (hep-th/9604148), Zacatecas, Jan. 1996.
41. C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, McGraw-Hill, New York (1980).

42. N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, Wiley, New York (1980).
43. N. Cabbibo and E. Ferrari, *Nuovo Cim.* **23** (1962) 1147; M. Y. Han and L. C. Biedenharn, *Nuovo Cim. A* **2** (1971) 544; R. Mignani, *Phys. Rev. D* **13** (1976) 2437.
44. M. A. Defaria-Rosa, E. Recami, and W. A. Rodrigues, Jr., *Phys. Lett. B* **173** (1986) 233; **188** (1987) 511E; M. A. Defaria-Rosa *et al., Phys. Lett. B* **220** (1989) 195.
45. M. A. Markov, On the difference between muon and electron masses, Preprint JINR D-1345, Dubna: JINR, 1963.
46. J. Brana and K. Ljolje, *Fizika* **12** (1980) 287.
47. A. O. Barut and G. Ziino, *Mod. Phys. Lett. A* **8** (1993) 1011; G. Ziino, *Int. J. Mod. Phys. A* **11** (1996) 2081.
48. P. M. Lavrov, *J. Phys. A* **18** (1985) 3455.
49. M. V. Ostrogradsky, *Mem. Acad. St. Petersbourg* **6** (1850) 385 [cited in ref. 50].
50. C. Grosse-Knetter, *Phys. Rev. D* **49** (1994) 6709; M. S. Rashid and S. S. Khalil, Hamiltonian description of higher order Lagrangians, Preprint IC/93/420, Trieste, 1993.
51. D. V. Ahluwalia and M. Sawicki, *Phys. Rev. D* **47** (1993) 5161.
52. J. D. Jackson, *Classical Electrodynamics*, 2nd ed., Wiley, New York (1975)
53. F. J. Dyson, *Am. J. Phys.* **58** (1990) 209; N. Dombey, *Am. J. Phys.* **59** (1991) 85; R. W. Brehme, *Am. J. Phys.* **59** (1991) 85; J. L. Anderson, *Am. J. Phys.* **59** (1991) 86; I. E. Farquhar, *Am. J. Phys.* **59** (1991) 87; R. J. Hughes, *Am. J. Phys.* **60** (1992) 301; C. R. Lee; *Phys. Lett. A* **148** (1990) 146; I. E. Farquhar, *Phys. Lett. A.* **151** (1991) 203; A. Vaidya and C. Farina, *Phys. Lett. A.* **153** (1991) 265.
54. S. Tanimura, *Ann. Phys.* (NY) **220** (1992) 229.
55. D. Saad, L. P. Horwitz, and R. I. Arshansky, *Found. Phys.* **19** (1989) 1126; M. C. Land, N. Shnerb, and L. P. Horwitz, Feynman's proof of the Maxwell equations as a way beyond the Standard Model, Preprint TAUP-2076-93 (hep-th/9308003), Tel Aviv, 1993.
56. A. Gersten, Conserved currents of the Maxwell equations in the presence of electric and magnetic sources, Preprint CERN-TH.4687/87, CERN, Geneva, 1987; Conserved currents of the Maxwell equations with electric and magnetic sources, Preprint CERN-TH.4688/87, CERN, Geneva, 1987.
57. A. Bandyopadhyay, *Int. J. Theor. Phys.* **32** (1993) 1563.
58. R. P. Malik and T. Pradhan, *Z. Phys. C* **28** (1985) 525.
59. B. Sarel and L. P. Horwitz, A chiral spin theory in the framework of an invariant evolution parameter formalism, Preprint TAUP-2280-95; M. C. Land and L. P. Horwitz, Off-shell quantum electrodynamics, Preprint TAUP-2227-95, Tel Aviv, 1995.
60. J. R. Oppenheimer, *Phys. Rev.* **38** (1931) 725.
61. E. Gianetto, *Lett. Nuovo Cim.* **44** (1985) 140.
62. M. Moshinsky and A. Del Sol, *Can. J. Phys.* **72** (1994) 453.
63. J.-M. Levy-Leblond, *Comm. Math. Phys.* **6** (1967) 286; M. Le Bellac and J.-M. Levy-Leblond, *Nuovo Cim. B* **14** (1973) 217.
64. F. S. Crawford, *Am. J. Phys.* **60** (1992) 109; **61** (1993) 472; A. Horzela, E. Kapuścik, and C. A. Uzes, *Am. J. Phys.* **61** (1993) 471.
65. F. J. Belinfante, *Physica* **6** (1939) 887.
66. H. C. Ohanian, *Am. J. Phys.* **54** (1986) 500.
67. A. E. Chubykalo and R. Smirnov-Rueda, *Phys. Rev. E* **53** (1996) 5373; **55** (1997) 3793E.
68. F. J. Belinfante, *Phys. Rev.* **76** (1949) 226; D. I. Caplan and F. J. Belinfante, *Phys. Rev.* **85** (1952) 721.